Term Rewriting Systems

Summer Semester 2018

Exercise Sheet 9 – Termination

13th June 2018

Prof. Dr.-Ing. Franz Baader, Dipl.-Math. Francesco Kriegel

Exercises 2 and 8 will not be discussed during the tutorial. Instead, solutions can be handed in at the next tutorial or, alternatively, sent to `francesco.kriegel@tu-dresden.de`. Individual feedback will then be provided.

Exercise 9.1 For each of the following statements, check whether it holds true.

(a) $f(x) \trianglerighteq_{\text{emb}} a$

(b) $f(b) \trianglerighteq_{\text{emb}} a$

(c) $g(g(x,y), g(a, f(z))) \trianglerighteq_{\text{emb}} g(y, g(a,z))$

★ **Exercise 9.2** Prove the following statements.

(a) The reduction relation $\xrightarrow{*}_{R_{\text{emb}}}$ induced by the rewrite system

$$R_{\text{emb}} := \{\, f(x_1, \ldots, x_n) \to x_i \mid n \geq 1,\ f \in \Sigma^{(n)},\ 1 \leq i \leq n \,\}$$

coincides with the homeomorphic embedding $\trianglerighteq_{\text{emb}}$, that is, for all terms s and t, it holds true that $s \xrightarrow{*}_{R_{\text{emb}}} t$ if, and only if, $s \trianglerighteq_{\text{emb}} t$.

(b) The homeomorphic embedding $\trianglerighteq_{\text{emb}}$ is a partial order.

(c) The homeomorphic embedding $\trianglerighteq_{\text{emb}}$ is well-founded. (Prove this without using Kruskal's Theorem.)

Exercise 9.3 In the proof of Theorem 5.4.8 in *Term Rewriting and All That* we have used the fact that the homeomorphic embedding $\trianglerighteq_{\text{emb}}$ is a well-partial-order. Why is it not sufficient for the proof just to know that $\trianglerighteq_{\text{emb}}$ is a well-founded order?

Exercise 9.4 Demonstrate that the termination of the following term rewrite system cannot be proved using a lexicographic path order.

$$R := \{f(f(x)) \to g(x),\ g(g(x)) \to f(x)\}$$

Exercise 9.5 Show termination of the following term rewrite system using a lexicographic path order.

$$R := \{s(x) + (y + z) \to x + (s(s(y)) * z),\ s(x_1) + (x_2 + (x_3 + x_4)) \to x_1 + (x_2 + (x_3 + x_4))\}$$

Exercise 9.6 Let Σ be a finite signature with at least one constant symbol, $>$ a strict partial order on Σ, and $>_{\text{lpo}}$ the lexicographic path order induced by $>$. Prove the following claim: if $>$ is a total order on Σ, then $>_{\text{lpo}}$ is total on ground terms.

Exercise 9.7 Prove the following claim: if $>$ is a reduction order on $T(\Sigma, V)$ that is total on ground terms, then $>$ satisfies the subterm property on ground terms, i.e., for each ground term t and for each position $p \in \mathcal{P}\mathrm{os}(t) \setminus \{\epsilon\}$, it holds true that $t > t|_p$.

★ **Exercise 9.8** Prove the first part of Proposition 5.4.16 in *Term Rewriting and All That*.

> Let Σ be a finite signature, $s, t \in T(\Sigma, V)$, and R a finite term rewriting system over $T(\Sigma, V)$. For a given lexicographic path order, $s >_{\mathsf{lpo}} t$ can be decided in time polynomial in the size of s, t.

Hint. Begin with showing that the condition

$$s >_{\mathsf{lpo}} t_j \text{ for all } j \text{ with } 1 \leq j \leq n$$

in (LPO2c) can be replaced with

$$s >_{\mathsf{lpo}} t_j \text{ for all } j \text{ with } i \leq j \leq n \text{ for } i \text{ such that } s_1 = t_1, \ldots, s_{i-1} = t_{i-1}, \text{ and } s_i >_{\mathsf{lpo}} t_i.$$

Use this modified condition to prove that the question whether $s >_{\mathsf{lpo}} t$ holds true can be decided in time $\mathcal{O}(|s| \cdot |t|)$.