Term Rewriting Systems

Exercise Sheet 13 - Gröbner Bases and Buchberger’s Algorithm

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Exercise 13.1 Let $I$ be an ideal of $K[X_1,\ldots,X_n]$. Show that $I$ contains the zero polynomial, and that $\equiv_I$ is a congruence relation on $K[X_1,\ldots,X_n]$.

Exercise 13.2 Let $f_1,\ldots,f_k \in K[X_1,\ldots,X_n]$. Show that $\langle f_1,\ldots,f_k \rangle$ is the smallest ideal of $K[X_1,\ldots,X_n]$ that contains $f_1,\ldots,f_k$.

Exercise 13.3 Show that the following binary relation $\succ$ on $M_n$, which is defined in Example 8.2.4 in Term Rewriting and All That, is an admissible total order.

\[ X_1^{k_1} \cdots X_n^{k_n} \succ X_1^{\ell_1} \cdots X_n^{\ell_n} \text{ if } \sum_{i=1}^n k_i > \sum_{i=1}^n \ell_i \]

\[ \text{or } \sum_{i=1}^n k_i = \sum_{i=1}^n \ell_i \text{ and } (k_1,\ldots,k_n) >_{\text{lex}}^n (\ell_1,\ldots,\ell_n) \]

Exercise 13.4 Give an example for an admissible order on $M_n$ different from $\succ$ in Exercise 3.

Exercise 13.5 Fix some well-founded total order $\succ$ on $M_n$. Show that the following implication is valid.

\[ m_1 \prec m_2 \Rightarrow m \cdot m_1 \prec m \cdot m_2 \quad \text{implies} \quad m_1 \mid m_2 \Rightarrow m_1 \preceq m_2 \]

Exercise 13.6 Let $G$ be a finite set of polynomials with head coefficient 1, and let $I := \langle G \rangle$ be the ideal generated by $G$. Show the following.

(a) If $f, g \in G$, then $S(f,g) \in I$.

(b) If $f \in I$ and $f \not\succ_G h$, then $h \in I$.

Exercise 13.7 Apply Buchberger’s algorithm to $F := \{X_1^2 \cdot X_2 - X_1^2, X_1 \cdot X_2^2 - X_2^3\}$. 