Term Rewriting Systems

Summer Semester 2018

Exercise Sheet 13 – Gröbner Bases and Buchberger's Algorithm

11th July 2018

Prof. Dr.-Ing. Franz Baader, Dipl.-Math. Francesco Kriegel

Exercise 13.1 Let J be an ideal of $K[X_1, \ldots, X_n]$. Show that J contains the zero polynomial, and that $\equiv_J$ is a congruence relation on $K[X_1, \ldots, X_n]$.

Exercise 13.2 Let $f_1, \ldots, f_k \in K[X_1, \ldots, X_n]$. Show that $\langle f_1, \ldots, f_k \rangle$ is the smallest ideal of $K[X_1, \ldots, X_n]$ that contains $f_1, \ldots, f_k$.

Exercise 13.3 Show that the following binary relation $\succ$ on M_n, which is defined in Example 8.2.4 in *Term Rewriting and All That*, is an admissible total order.

$$X_1^{k_1} \ldots X_n^{k_n} \succ X_1^{\ell_1} \ldots X_n^{\ell_n} \quad \text{if} \quad \sum_{i=1}^{n} k_i > \sum_{i=1}^{n} \ell_i$$
$$\text{or} \quad \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \ell_i \text{ and } (k_1, \ldots, k_n) >_{\mathsf{lex}}^{n} (\ell_1, \ldots, \ell_n)$$

Exercise 13.4 Give an example for an admissible order on M_n different from $\succ$ in Exercise 3.

Exercise 13.5 Fix some well-founded total order $\succ$ on M_n. Show that the following implication is valid.

$$m_1 \prec m_2 \Rightarrow m \cdot m_1 \prec m \cdot m_2 \quad \text{implies} \quad m_1 \mid m_2 \Rightarrow m_1 \preceq m_2$$

Exercise 13.6 Let G be a finite set of polynomials with head coefficient 1, and let $J := \langle G \rangle$ be the ideal generated by G. Show the following.

(a) If $f, g \in G$, then $S(f, g) \in J$.

(b) If $f \in J$ and $f \xrightarrow{*}_G h$, then $h \in J$.

Exercise 13.7 Apply Buchberger's algorithm to $F := \{X_1^2 \cdot X_2 - X_1^2,\ X_1 \cdot X_2^2 - X_2^2\}$.