Unification

basic definitions and results

Definition

A unifier of the terms s, t is a substitution σ such that $\sigma(s) = \sigma(t)$.

A substitution σ is more general than a substitution σ' if there is a substitution δ such that

$$\sigma' = \delta\sigma.$$

In this case we write $\sigma \lesssim \sigma'$, and say that σ' is an instance of σ.

Definition

The unifier σ of the terms s, t is a most general unifier (mgu) iff every unifier of s, t is an instance of σ.

We will show: if s, t have a unifier, then they have an mgu, and this mgu can effectively be computed.

Unification

in more detail

Definition

A unification problem is a finite set of equations

$$S = \{s_1 =^? t_1, \ldots, s_n =^? t_n\}.$$

A unifier or solution of S is a substitution σ such that $\sigma(s_i) = \sigma(t_i)$ for $i = 1, \ldots, n$.

$\mathcal{U}(S)$ denotes the set of all unifiers of S.

A substitution σ is a most general unifier (mgu) of S if

- $\sigma \in \mathcal{U}(S)$ and
- $\forall \sigma' \in \mathcal{U}(S).\ \sigma \lesssim \sigma'$.

Examples

some typical situations

$\{f(x) =^? f(a)\}$ has $\{x \mapsto a\}$ as a unifier

$\{x =^? f(y)\}$ has $\{x \mapsto f(y)\},\ \{x \mapsto f(a), y \mapsto a\}, \ldots$ as unifiers

$\{x =^? f(x)\}$ does not have a unifier

$\{f(x) =^? g(y)\}$ does not have a unifier

Unification

by transformation

Idea:

Transform set of equations into solved form, from which the mgu can be obtained immediately.

Similar to Gaussian elimination in linear algebra:

$$\begin{array}{|rcl|}\hline x+3y &=& 0\\ 2x+8y &=& 2z\\\hline\end{array} \rightsquigarrow \begin{array}{|rcl|}\hline x+3y &=& 0\\ 2y &=& 2z\\\hline\end{array} \rightsquigarrow \begin{array}{|rcl|}\hline x+3y &=& 0\\ y &=& z\\\hline\end{array}$$

$$\rightsquigarrow \begin{array}{|rcl|}\hline x+3z &=& 0\\ y &=& z\\\hline\end{array} \rightsquigarrow \begin{array}{|rcl|}\hline x &=& -3z\\ y &=& z\\\hline\end{array}$$

Definition

A unification problem

$$S = \{x_1 =^? t_1, \ldots, x_n =^? t_n\}$$

is in solved form if the x_i are pairwise distinct variables, none of which occurs in any of the t_i.

In this case we define

$$\vec{S} := \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}.$$

Lemma

If S is in solved form then $\vec{S}$ is an mgu of S.

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The transformation rules

Delete	$\{t =^? t\} \uplus S$	$\Longrightarrow$	S
Decompose	$\{f(t_1, \ldots, t_n) =^? f(u_1, \ldots, u_n)\} \uplus S$	$\Longrightarrow$	$\{t_1 =^? u_1, \ldots, t_n =^? u_n\} \cup S$
Orient	$\{t =^? x\} \uplus S$	$\Longrightarrow$	$\{x =^? t\} \cup S$ if $t \notin V$
Eliminate	$\{x =^? t\} \uplus S$	$\Longrightarrow$	$\{x =^? t\} \cup \{x \mapsto t\}(S)$ if $x \in \mathcal{V}ar(S) - \mathcal{V}ar(t)$

Example

$$\{x =^? f(a),\ g(x,x) =^? g(x,y)\} \quad \Longrightarrow_{\text{Eliminate}}$$

$$\{x =^? f(a),\ g(f(a),f(a)) =^? g(f(a),y)\} \quad \Longrightarrow_{\text{Decompose}}$$

$$\{x =^? f(a),\ f(a) =^? f(a),\ f(a) =^? y\} \quad \Longrightarrow_{\text{Delete}}$$

$$\{x =^? f(a),\ f(a) =^? y\} \quad \Longrightarrow_{\text{Orient}}$$

$$\{x =^? f(a),\ y =^? f(a)\}$$

$Unify(S) \quad = $

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while there is some T such that S ⟹ T do S := T;
if S is in solved form then return S⃗ else fail.
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Lemma (termination, soundness, completeness)

1. *Unify* terminates for all inputs.
2. If $S \Longrightarrow T$ then $\mathcal{U}(S) = \mathcal{U}(T)$.
3. If $Unify(S)$ returns σ, then σ is an mgu of S.
4. If $Unify(S)$ fails, then the final set of equations contains an equation of the form
   (a) $x =^? t$ with $x \in \mathcal{V}ar(t), x \neq t$,
   (b) $f(\cdots) =^? g(\cdots)$ with $f \neq g$.
5. If $Unify(S)$ fails, then S has no solution.

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Theorem

The function *Unif* decides, for any input unification problem S, whether it has a solution or not.

If S has a solution, then *Unif* computes an mgu of S.

Complexity:

- The worst-case complexity of this unification algorithm is exponential (both time and space).
- There exists a linear time unification algorithm.

Matching

a special case of unification

Given terms l, s, find a substitution σ such that $\sigma(l) = s$.

The substitution σ is called a matcher of the matching problem $l \lesssim^? s$.

Reduce matching to unification: regard all variables in s as constants,

by introducing a new constant c_x for each variable x.

Example

The matching problem $f(x, y) \lesssim^? f(g(z), x)$

becomes the unification problem $\{f(x, y) =^? f(g(c_z), c_x)\}$.

The mgu $\{x \mapsto g(c_z), y \mapsto c_x\}$

becomes the matcher $\{x \mapsto g(z), y \mapsto x\}$.

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