Exercise 4.1 Prove or refute the following claim: If an $\mathcal{ALC}$-concept description $C$ is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $T$, then for all $n \geq 1$ there is a finite model $I_n$ of $T$ such that $|C^{I_n}| \geq n$.

Does the claim hold true if the condition "$|C^{I_n}| \geq n$" is replaced by "$|C^{I_n}| = n$"?

Exercise 4.2 Prove or refute the following claim: Given an $\mathcal{ALC}$-concept description $C$ and an $\mathcal{ALC}$-TBox $T$, if $I$ is an interpretation and $J$ its filtration w.r.t. $\text{sub}(C) \cup \text{sub}(T)$ (see Definition 3.14), then the relation $\rho := \{(d, [d]) \mid d \in \Delta^I\}$ is a bisimulation between $I$ and $J$.

Hint. If the above relation $\rho$ were a bisimulation, why do we have to explicitly prove Lemma 3.15 in the lecture? Wouldn't Lemma 3.15 then be a consequence of Theorem 3.2?

Exercise 4.3 We consider bisimulations between an interpretation $I$ and itself, which are called bisimulations on $I$. For two elements $d, e \in \Delta^I$, we write $d \approx_I e$ if they are bisimilar, i.e., if there is a bisimulation $\rho$ on $I$ such that $d \rho e$.

(a) Show that $\approx_I$ is an equivalence relation on $\Delta^I$.

(b) Show that $\approx_I$ is a bisimulation on $I$.

(c) Show that, for finite interpretations $I$, the relation $\approx_I$ can be computed in time polynomial in the cardinality of $I$ (if the signature is finite as well).

Consider the interpretation $J$ that is defined like the filtration (Definition 3.14), but with $\approx_I$ instead of $\simeq$.

(d) Show that $\rho := \{(d, [d]_{\approx_I}) \mid d \in \Delta^I\}$ is a bisimulation between $I$ and $J$.

(e) Show that, if $I$ is a model of an $\mathcal{ALC}$-concept description $C$ w.r.t. an $\mathcal{ALC}$-TBox $T$, then so is $J$.

(f) Why can we not use the previous result to show the finite model property for $\mathcal{ALC}$?

Exercise 4.4 For the following interpretation $I$, draw the unraveling of $I$ at $d$ up to depth 5, i.e., restricted to $d$-paths of length at most 5 (see Definition 3.21):

```
\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (s) at (-1,-1) {$s$};
  \node (r) at (-1,-2) {$r$};
  \node (f) at (-2,-1) {$f$};
  \node (e) at (-2,-2) {$e$};

  \draw[->] (s) -- (A);
  \draw[->] (r) -- (A);
  \draw[->] (f) -- (r);
  \draw[->] (e) -- (r);
  \draw[->] (s) -- (f);
  \draw[->] (e) -- (A);

\end{tikzpicture}
\end{center}
```
Exercise 4.5  Prove or refute the following claim: If $\mathcal{K}$ is an $\mathcal{ALC}$-knowledge base and $C$ is an $\mathcal{ALC}$-concept description such that $C$ is satisfiable w.r.t. $\mathcal{K}$, then $C$ has a tree model w.r.t. $\mathcal{K}$. 