Exercise 8.1 Prove that the tableau algorithm for $\mathcal{ALCN}$ is sound and complete.

Exercise 8.2 We extend the tableau algorithm from $\mathcal{ALCN}$ to $\mathcal{ALCQ}$ by modifying the $\geq$-rule and the $\leq$-rule as follows:

<table>
<thead>
<tr>
<th>The $\geq$-rule</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Condition:</strong> $\mathcal{A}$ contains $a:(\geq n \cdot r.C)$, but there are no $n$ distinct individuals $b_1, \ldots, b_n$ with ${(a, b_i) : r, b_i : C \mid 1 \leq i \leq n} \subseteq \mathcal{A}$, and $a$ is not blocked</td>
</tr>
<tr>
<td><strong>Action:</strong> $\mathcal{A} \rightarrow \mathcal{A} \cup {(a, d_i) : r, d_i : C \mid 1 \leq i \leq n} \cup {(d_i, d_j) : 1 \leq i &lt; j \leq n}$, where $d_1, \ldots, d_n$ are new individual names</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>The $\leq$-rule</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Condition:</strong> $\mathcal{A}$ contains $a:(\leq n \cdot r.C)$, and there are $n+1$ distinct individuals $b_0, \ldots, b_n$ with ${(a, b_i) : r, b_i : C \mid 0 \leq i \leq n} \subseteq \mathcal{A}$</td>
</tr>
<tr>
<td><strong>Action:</strong> $\mathcal{A} \rightarrow \text{prune}(\mathcal{A}, b_j) [b_j \mapsto b_i] \cup {b_i = b_j}$ for $i \neq j$ such that, if $b_j$ is a root individual, then so is $b_i$</td>
</tr>
</tbody>
</table>

For the knowledge base

\[
\{ \{ C \sqsubseteq E \}, \{ a:(\leq 1 \cdot r.(D \sqcap E)), (a, b) : r, b : C \sqcap D, (a, c) : r, c : D \sqcap E, c : \neg C \} \},
\]

determine whether it is consistent, and whether the proposed algorithm detects this.

Exercise 8.3 Let $\mathcal{T}$ be an acyclic TBox in NNF, and let $\mathcal{T}^E$ be obtained from $\mathcal{T}$ by replacing each concept definition $A \equiv C$ with the concept inclusion $A \sqsubseteq C$.

Prove that every concept name is satisfiable w.r.t. $\mathcal{T}$ if, and only if, it is satisfiable w.r.t. $\mathcal{T}^E$. Does this also hold true for the acyclic TBox $\{ A \equiv C \sqcap \neg B, B \equiv P, C \equiv P \}$?

Exercise 8.4 Use the $\mathcal{ALC}$-Worlds algorithm to decide satisfiability of the concept name $A_0$ w.r.t. the following simple TBox:

\[
\begin{align*}
A_0 & \equiv A_1 \sqcap A_2, \\
A_1 & \equiv \exists r.A_3, \\
A_2 & \equiv A_4 \sqcap A_5, \\
A_3 & \equiv P, \\
A_4 & \equiv \exists r.A_6, \\
A_5 & \equiv A_7 \sqcap A_8, \\
A_6 & \equiv Q, \\
A_7 & \equiv \forall r.A_4, \\
A_8 & \equiv \forall r.A_9, \\
A_9 & \equiv \forall r.A_{10}, \\
A_{10} & \equiv \neg P
\end{align*}
\]

Draw the recursion tree of a successful run and of an unsuccessful run. Does the algorithm return a positive or a negative result on this input?