Exercise 3.1 Let $\Sigma$ be an alphabet and $(M, \circ, 1)$ a monoid. Prove that every function $f : \Sigma \to M$ can be uniquely extended to a homomorphism $\phi : \Sigma^* \to M$.

*Hint.* If $f : A \to C$ and $g : B \to C$ are functions such that $A \subseteq B$, then we say that $g$ extends $f$ if $g(a) = f(a)$ for each $a \in A$.

Exercise 3.2 Consider the monoids $M_i := (\{1, a, b\}, \circ_i, 1)$ for $i \in \{1, 2\}$, where $\circ_1$ is given by the following table:

<table>
<thead>
<tr>
<th>$\circ_1$</th>
<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

and $x \circ_2 y := y \circ_1 x$ for all $x, y \in \{1, a, b\}$.

For each $i \in \{1, 2\}$, find a regular language $L_i \subseteq \{a, b\}^*$ such that $M_i$ is the syntactic monoid of $L_i$, or prove that no such language exists.

Exercise 3.3 Let $\Sigma := \{a, b\}$, $M := \{0, 1, 2\}$, and let $\circ : M \times M \to M$ be defined as $x \circ y := (x + y) \mod 3$. We define mappings $\phi, \phi' : \Sigma^* \to M$ by setting $\phi(w) := |w| \mod 3$ and $\phi'(w) := |w|_a \mod 3$, where $|w|$ denotes the length of $w$ and $|w|_a$ the number of occurrences of the symbol $a$ in $w$.

(a) Show that both $\phi$ and $\phi'$ are homomorphisms from $(\Sigma^*, \cdot, \varepsilon)$ into $(M, \circ, 0)$.

(b) For each of the languages $\phi^{-1}(\{0, 2\}), \phi'^{-1}(\{1\})$ and $(\phi')^{-1}(\{1\})$ devise a finite automaton that recognises the language.

Exercise 3.4 Let $\Sigma$ be an alphabet, $L \subseteq \Sigma^*$ a language, and $(M, \circ, 1)$ a monoid. Prove that $L$ is accepted by $(M, \circ, 1)$ if, and only if, $\overline{L}$ is also accepted by $(M, \circ, 1)$.

Exercise 3.5 Determine the syntactic monoid of the language described by $a^*ba^*$. 
**Exercise 3.6** Let \( L \subseteq \Sigma^* \), and \( \approx \) be an equivalence relation on \( \Sigma^* \). Consider the following property:

For all \( u, v \in \Sigma^* \), if \( u \in L \) and \( u \approx v \), then \( v \in L \). \((\ast)\)

(a) Prove that the syntactical congruence \( \sim_L \) has property \((\ast)\).

*Hint.* The proof of Corollary 1.13 from the lecture depends on this fact.

(b) Show that \( \sim_L \) is the coarsest congruence relation with property \((\ast)\).

*Hint.* An equivalence relation \( \approx_2 \) is *coarser* than \( \approx_1 \) if, for every \( x, y \), \( x \approx_1 y \) implies \( x \approx_2 y \). (In particular, \( \approx_2 \) has at most as many equivalence classes as \( \approx_1 \).)

(c) Show that the Nerode right congruence \( \rho_L \) is the coarsest right congruence with property \((\ast)\).