TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Automata and Logic

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Exercise Sheet 3

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Regular Languages, Finite Monoids, and Logical Formulae

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Exercise 3.1 Let Σ be a finite alphabet and $(M, \circ, 1)$ a monoid. Prove that every function $f\colon \Sigma \to M$ can be uniquely extended to a homomorphism $\phi\colon \Sigma^* \to M$.

Hint. If $f\colon A \to C$ and $g\colon B \to C$ are functions such that $A \subseteq B$, then we say that g *extends* f if $g(a) = f(a)$ for each $a \in A$.

Exercise 3.2 Consider the monoids $M_i := (\{1, a, b\}, \circ_i, 1)$ for $i \in \{1, 2\}$, where $\circ_1$ is given by the following table:

$\circ_1$	1	a	b
1	1	a	b
a	a	a	b
b	b	a	b

and $x \circ_2 y := y \circ_1 x$ for all $x, y \in \{1, a, b\}$.

For each $i \in \{1, 2\}$, find a regular language $L_i \subseteq \{a, b\}^*$ such that M_i is the syntactic monoid of L_i, or prove that no such language exists.

Exercise 3.3 Let $\Sigma := \{a, b\}$, $M := \{0, 1, 2\}$, and let $\circ\colon M \times M \to M$ be defined as $x \circ y := (x + y) \bmod 3$. We define mappings $\phi, \phi'\colon \Sigma^* \to M$ by setting $\phi(w) := |w| \bmod 3$ and $\phi'(w) := |w|_a \bmod 3$, where $|w|$ denotes the *length* of w and $|w|_a$ the number of occurrences of the symbol a in w.

(a) Show that both ϕ and ϕ' are homomorphisms from $(\Sigma^*, \cdot, \varepsilon)$ into $(M, \circ, 0)$.

(b) For each of the languages $\phi^{-1}(\{0, 2\})$, $\phi^{-1}(\{1\})$ and $(\phi')^{-1}(\{1\})$ devise a finite automaton that recognises the language.

Exercise 3.4 Let Σ be a finite alphabet, $L \subseteq \Sigma^*$ a language, and $(M, \circ, 1)$ a monoid. Prove that L is accepted by $(M, \circ, 1)$ if, and only if, $\overline{L}$ is also accepted by $(M, \circ, 1)$.

Exercise 3.5 Determine the syntactic monoid of the language described by a^*ba^*.

Exercise 3.6 Let $L \subseteq \Sigma^*$ for some finite alphabet Σ, and $\approx$ be an equivalence relation on Σ^*. Consider the following property:

$$\text{For all } u, v \in \Sigma^*, \text{ if } u \in L \text{ and } u \approx v, \text{ then } v \in L. \tag{$*$}$$

(a) Prove that the syntactical congruence $\sim_L$ has property $(*)$.

Hint. The proof of Corollary 1.13 from the lecture depends on this fact.

(b) Show that $\sim_L$ is the coarsest congruence relation with property $(*)$.

Hint. An equivalence relation $\approx_2$ is *coarser* than $\approx_1$ if, for every x, y, $x \approx_1 y$ implies $x \approx_2 y$. (In particular, $\approx_2$ has at most as many equivalence classes as $\approx_1$.)

(c) Show that the Nerode right congruence ρ_L is the coarsest right congruence with property $(*)$.