Exercise 4.1 Show that any submonoid of a finite group is also a group.

Exercise 4.2 Let $V$ be an M-variety. Show that $L(V)_\Sigma$ is closed under union without using Thm. 1.22 from the lecture.

Exercise 4.3 Let $\Sigma$ be an alphabet. Prove or refute the following claims:

(a) Every regular language $L \subseteq \Sigma^*$ is accepted by its syntactic monoid.

(b) If $L \subseteq \Sigma^*$ is accepted by a finite group, then the syntactic monoid of $L$ is a finite group.

(c) For every regular language $L \subseteq \Sigma^*$, the syntactic monoid $M_L$ is the smallest monoid accepting $L$; i.e., for every monoid $M$ accepting $L$, we have $|M_L| \leq |M|$.

(d) For a word $w = a_1 \ldots a_n$, let $\overline{w}$ denote the mirror image of $w$, i.e., $\overline{w} = a_n \ldots a_1$. For a language $L \subseteq \Sigma^*$, we define $\overline{L} := \{\overline{w} \mid w \in L\}$. **Claim:** If the minimal automaton for $L$ has $n$ states, then the minimal automaton for $\overline{L}$ has also $n$ states.

Exercise 4.4 Let $L_1$ be the language over $\{a\}$ described by $a^+$, and let $L_2$ be the language over $\{a, b\}$ described by $(a + b)^*b(a + b)^*$.

(a) Is there a monoid that accepts both $L_1$ and $L_2$?

(b) Are the syntactic monoids of those languages isomorphic?

Exercise 4.5 Let $L_1$ and $L_2$ be two languages over the same alphabet $\Sigma$ that are accepted by the same monoid $(M, \circ, 1)$. Prove or refute the following statements:

(a) $M$ accepts $L_1 \cap L_2$.

(b) $M$ accepts $L_1 \cup L_2$.

(c) $M$ accepts $L_1 \cdot L_2$.

Exercise 4.6 Let $V$ be the M-variety of all commutative finite groups. Show that there exists a language $L \subseteq \{a\}^*$ such that $L \in L(V)_{\{a\}}$ but $L \notin L(V)_{\{a,b\}}$. 

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