Automata and Logic

Exercise Sheet 5

Regular Languages, Finite Monoids, and Logical Formulae

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Exercise 5.1 Prove or refute the following claim. There is a language $L \subseteq \{a, b\}^*$ such that its syntactic semigroup $S_L$ and its syntactic monoid $M_L$ are isomorphic.

Exercise 5.2 (a) For each of the following words over the alphabet $\{0, 1\}^k$, give a corresponding interpretation over the predicate symbols $P_1, \ldots, P_k$ as discussed in the lecture.

- $k = 2$: $(1, 1), (1, 1), (0, 1), (1, 0)$
- $k = 3$: $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$
- $k = 3$: $(1, 1, 0), (1, 0, 1), (1, 1, 1), (1, 1, 0)$

(b) Describe all interpretations that correspond to words of the language $L((0, 1) \cdot (1, 0))^+ \subseteq (\{0, 1\}^2)^+$.

Exercise 5.3 Let $\Sigma := \{a, b\}$. For each of the following regular expressions $r_i$, give a first-order formula $\phi_i$ such that $L(r_i) = L(\phi_i)$.

- (a) $r_1 := \Sigma^*$
- (b) $r_2 := \varepsilon$
- (c) $r_3 := (a b a^*)^*$
- (d) $r_4 := a^* b^* + b^* a^*$
- (e) $r_5 := (aaa \cdot \Sigma^*) + b^*$

Exercise 5.4 Let $(S, \circ)$ be a finite semigroup, $m \in S$, and $i, k, \ell \in \mathbb{N} \setminus \{0\}$ defined as in the proof of Proposition 2.4. Show that if $k$ is minimal with respect to the property described in the proof, then

$$(\{m^i, \ldots, m^{i+k-1}\}, \circ, m^\ell)$$

is a group. Is it still a group if $k$ is not minimal?
Exercise 5.5 Let $\Sigma \coloneqq \{a, b\}$ and $L_1, L_2$ be the languages accepted by the automata displayed below.

Use the proof of Corollary 2.10 to show that $L_1 \not\in (B_0)_\Sigma$ and $L_2 \in (B_0)_\Sigma$. Moreover, represent $L_2$ as a Boolean combination of languages from the set

$$\{u\Sigma^* \mid u \in \Sigma^*\} \cup \{\Sigma^* u \mid u \in \Sigma^*\}.$$

Exercise 5.6 Prove or refute the following claims.

(a) For every alphabet $\Sigma$ and any word $w \in \Sigma^*$, we have $\{w\} \in (B_0)_\Sigma$.

(b) For every two alphabets $\Sigma$ and $\Sigma'$ with $\Sigma \subseteq \Sigma'$, and every language $L \subseteq \Sigma^*$, we have: if $L \in (B_0)_\Sigma$, then $L \in (B_0)_{\Sigma'}$.

(c) Let $(M, \circ, 1)$ be a monoid, where 1 is the only idempotent element of $M$. Then $(M, \circ, 1)$ is a group.

(d) Let $(S, \circ)$ be a semigroup with $e \in S$ being idempotent. Then $(eSe, \circ, e)$ is the largest submonoid of $S$ with $e$ as unit element.

(e) Let $(S, \circ) \in \hat{D}$. If there exists an element $s \in S$ such that $(S, \circ, s)$ is a monoid, then $|S| = 1$. 
