TECHNISCHE UNIVERSITÄT DRESDEN

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Automata and Logic

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Generalized-Definite Languages and Quantifier-Free Formulae, Star-Free Languages

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Exercise 6.1 Let V be the class of all finite semigroups S such that, for any idempotent element $e \in S$, we have $Se = e$. Show that V is an S-variety ultimately defined by

$$(yx^n = x^n)_{n \geq 1}.$$

Exercise 6.2 Let $\Sigma := \{a, b, c, d\}$.

(a) For $L \subseteq \Sigma^*$ with

$$\begin{aligned} L := \{\, w \mid w \in \Sigma^* \text{ and } w \text{ starts with } a \text{ or } b \,\} \\ \cap \{\, w \mid w \in \Sigma^*,\ |w| \geq 3, \text{ and } w \text{ starts and ends with the same symbol} \,\}, \end{aligned}$$

give a quantifier-free formula ϕ using the signature $\{Q_a, Q_b, Q_c, Q_d, <, \mathsf{min}, \mathsf{max}, s, p\}$ such that $L(\phi) = L$.

(b) Let

$$\phi := \neg(\neg Q_a(s(s(p(s(\mathsf{min}))))) \vee (s(\mathsf{min}) < p(p(\mathsf{max})))).$$

Use the method described in the proof of Proposition 2.11 to describe $L(\phi)$ as a Boolean combination of languages from the set $\{\, u\Sigma^*,\ \Sigma^* u \mid u \in \Sigma^* \,\}$.

Exercise 6.3 Let Σ be a finite alphabet. A language $L \subseteq \Sigma^*$ is called *definite* for Σ if there exists an $n \in \mathbb{N}$ such that we have for all $w \in L$:

$$\text{if } w = uv \text{ with } |u| = n, \text{ then } u\Sigma^* \subseteq L.$$

Show that $L \subseteq \Sigma^*$ is definite for Σ if, and only if, L is a Boolean combination of languages of the form $\{\, w\Sigma^* \mid w \in \Sigma^* \,\}$.

Exercise 6.4 Let $\Sigma := \{0, 1\}^k$. Show that the following statements are equivalent.

- L is definite for Σ.
- There exists a quantifier-free closed first-order formula ϕ over the signature $\{P_1, \ldots, P_k, <, \mathsf{min}, s\}$ with $L(\phi) = L \setminus \{\varepsilon\}$.

Exercise 6.5 Let Σ, Γ be two finite alphabets, and let $L \subseteq \Sigma^*$. Prove or refute the following claims.

(a) $L \in \mathsf{SF}_\Sigma \Rightarrow L \in \mathsf{SF}_{\Sigma \cup \Gamma}$

(b) $L \in \mathsf{SF}_{\Sigma \cup \Gamma} \Rightarrow L \in \mathsf{SF}_\Sigma$

Exercise 6.6 For $\Sigma := \{a, b\}$, check whether the following languages are star-free.

(a) $L_1 := (ab)^*$

(b) $L_2 := \{ w \mid |w|_a = 3k \text{ for some } k \in \mathbb{N} \}$

(c) $L_3 := a(aba)^*b$

For each L_i, use Proposition 3.6 or give a star-free description of the language.