Automata and Logic

Exercise Sheet 13

Automata on Infinite Trees
Tree-Automata and Logical Formulae

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Exercise 13.1 Let $\Sigma := \{a, b\}$ be an alphabet with two binary symbols and consider the language

$$L := \{ t \in T_\omega^\Sigma \mid \text{there is a path in } t \text{ containing the symbol } a \text{ only finitely often} \}.$$ 

(a) Is $L$ Rabin-recognizable?
(b) Is $L$ Büchi-recognizable?

Exercise 13.2 Let $\Sigma$ be a finite alphabet with arity function containing at least two binary symbols $f$ and $g$. Is the $\omega$-tree language $L := \{ f(t, t) \mid t \in T_\omega^\Sigma \}$ Büchi-recognizable? Justify your answer.

Exercise 13.3 Let $\Sigma$ be a finite alphabet of binary symbols. For each $\omega$-tree $t \in T_\omega^\Sigma$, we define an $\omega$-language $\text{path}(t) \subseteq \Sigma^\omega$ as follows:

$$\text{path}(t) := \{ \alpha \in \Sigma^\omega \mid \text{there is a path } i_0i_1i_2\ldots \text{ in } t \text{ such that } t(i_0i_1\ldots i_n) = \alpha(n) \text{ for all } n \in \mathbb{N} \}.$$ 

Furthermore, let

$$\text{path}(B) := \bigcup \{ \text{path}(t) \mid t \in B \}$$

for each $B \subseteq T_\omega^\Sigma$. For an $\omega$-language $L \subseteq \Sigma^\omega$, let

$$\text{tree}(L) := \{ t \in T_\omega^\Sigma \mid \text{path}(t) \subseteq L \}.$$ 

Prove or refute each of the following claims.

(a) $\text{tree}(\text{path}(B)) = B$
(b) $\text{path}(\text{tree}(L)) = L$
(c) If $B$ is Büchi-recognizable, then $\text{path}(B)$ is also Büchi-recognizable.
(d) If $\text{path}(B)$ is Büchi-recognizable, then $B$ is also Büchi-recognizable.
(e) If $L$ is Büchi-recognizable, then $\text{tree}(L)$ is also Büchi-recognizable.
(f) If $\text{tree}(L)$ is Büchi-recognizable, then $L$ is also Büchi-recognizable.
**Exercise 13.4** Let $\Sigma := \{0, 1\}$ and define

$$L := \{ t \in T_{\Sigma}^\omega \mid \text{for every path } p, \text{ if } p \text{ contains the symbol } 0, \text{ then } p \text{ contains the symbol } 1 \text{ only finitely often } \}.$$ 

Give an S2S-formula $\phi$ such that $L_\omega(\phi) = L$. 

**Exercise 13.5** Let $\Sigma := \{a, b\}$ and consider the Rabin-automaton

$$A := (\{q_0, q_1, q_2\}, \Sigma, \{q_2\}, \Delta, \{\{q_0, q_2\}, \{q_1\}\})$$

for which the transition assignment $\Delta$ is defined in the following way.

- $\Delta_a: q_0 \mapsto \{(q_1, q_1)\}$
- $q_1 \mapsto \{(q_1, q_1)\}$
- $q_2 \mapsto \{(q_0, q_1)\}$

Find an S2S-formula $\phi$ satisfying $L_\omega(A) = L_\omega(\phi)$. 

**Exercise 13.6** Consider the $\omega$-tree language $L$ over a finite alphabet $\Sigma$ that is defined as

$$L := \{ t \in T_{\Sigma}^\omega \mid t \text{ is a strict subtree of } t \}.$$ 

For each of the following claims, decide whether it holds true and justify your answer. 

(a) There is a tree $t \in L$ such that it contains itself only finitely often as a strict subtree. 

(b) $L$ is Büchi-recognizable. 

(c) $L$ is Rabin-recognizable. 

(d) There is a closed S2S-formula $\phi$ such that $L_\omega(\phi) = L$ is satisfied.