Algorithms for Computing Least Common Subsumers in General $\mathcal{FL}_0$-TBoxes

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Master's Thesis

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A Bottom-up Approach for Ontology Construction

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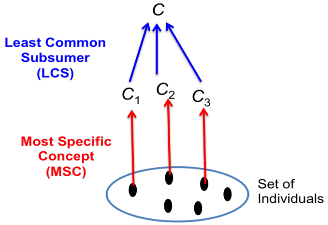

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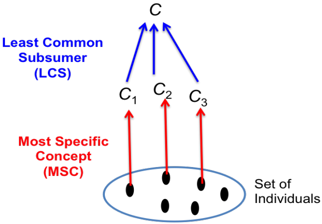

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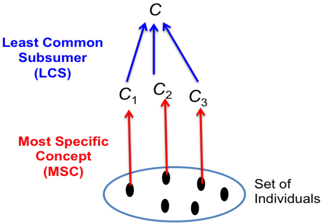

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   - **A decision procedure** to decide the existence of the lcs w.r.t. general $\mathcal{EL}$-TBoxes
   - **An algorithm** for computing least common subsumers in general $\mathcal{EL}$-TBoxes
3. How about $\mathcal{FL}_0$?
   - **No decision procedures** for the problem of the existence of the lcs w.r.t. general $\mathcal{FL}_0$-TBoxes.
   - **No algorithms** for computing least common subsumers in general $\mathcal{FL}_0$-TBoxes.

A Poet Composes A Poem

1. **Example 1: The lcs does not exist**
TBox $\mathcal{T}_1$:=

$$\begin{aligned}\{\texttt{Songwriter} &\sqsubseteq \texttt{Artist} \sqcap \forall \textit{composes}.\texttt{Song} \\ \texttt{Poet} &\sqsubseteq \texttt{Artist} \sqcap \forall \textit{composes}.\texttt{Poem} \\ \texttt{Song} &\sqsubseteq \texttt{Art} \sqcap \forall \textit{madeUpBy}.\texttt{Songwriter} \\ \texttt{Poem} &\sqsubseteq \texttt{Art} \sqcap \forall \textit{madeUpBy}.\texttt{Poet}\}\end{aligned}$$

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2. **Example 2: The lcs exists**
   $\mathcal{T}_2 := \mathcal{T}_1 \cup$

$$\begin{aligned}\{\texttt{Artist} &\sqsubseteq \forall \textit{composes}.\texttt{Art}\\ \texttt{Art} &\sqsubseteq \forall \textit{madeUpBy}.\texttt{Artist}\}\end{aligned}$$

   The lcs of `Songwriter` and `Poet` w.r.t. $\mathcal{T}_2$ is `Artist`.

Research Problems

Let C, D, E be $\mathcal{FL}_0$-concepts and $\mathcal{T}$ be a general $\mathcal{FL}_0$-TBox.

1. **Research Problem I (RP I):**

   Is concept E the lcs of C and D w.r.t. $\mathcal{T}$?

2. **Research Problem II (RP II):**

   Does the lcs of C and D w.r.t. $\mathcal{T}$ exist?

3. **Research Problem III (RP III):**

   If the lcs of C and D w.r.t. $\mathcal{T}$ exists, then what is the lcs?
   And how big is the size of the lcs?

Description Logic $\mathcal{FL}_0$

- N_C: **set of concept names** with $A \in N_C$ → Songwriter, Poet, Song, Poem, ...
- N_R: **set of role names** with $r \in N_R$ → writes, composes, madeUpBy, arranges, ...

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  - $\Delta^{\mathcal{I}}$: a non-empty domain.
    Here we define $\Delta^{\mathcal{I}} = N_R^*$
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- The mapping $\cdot^{\mathcal{I}}$ is extended to $\mathcal{FL}_0$-concepts

Syntax	Semantic
$\top$ (Top)	$\Delta^{\mathcal{I}}$
$C \sqcap D$ (Conjunction)	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$\forall r.C$ (Value Restriction)	$\{d \in \Delta^{\mathcal{I}} \mid e \in C^{\mathcal{I}} \text{ for all } (d,e) \in r^{\mathcal{I}}\}$

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Conventions:

- $\forall r_1.\forall r_2 \cdots \forall r_n A \equiv \forall w.A$, where $w = r_1 r_2 \ldots r_n \in N_R^*$.
- $A \equiv \forall \varepsilon.A$

$\mathcal{FL}_0$-TBoxes in CCNF and PANF

- A **(general) $\mathcal{FL}_0$ TBox** $\mathcal{T}$ is a finite set of General Concept Inclusions (GCIs) of the form of $C \sqsubseteq D$.
- $N_{C,\mathcal{T}}$: set of concept names occurring in $\mathcal{T}$.

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Normalization in $\mathcal{FL}_0$-TBoxes [Pensel,2015]

- A concept is in ***concept-conjunction-normal-form*** **(CCNF)** iff it is of the form
$$\forall w_1.A_1 \sqcap \ldots \sqcap \forall w_n.A_n,$$
where $A_i \in N_C$ and $w_i \in N_R^*$, for all $1 \leq i \leq n$.
- An $\mathcal{FL}_0$-TBox $\mathcal{T}$ is in ***plane-axiom-normal-form*** **(PANF)** iff
  - All left- and right-hand sides of all GCIs in $\mathcal{T}$ are in CCNF;
  - Every $\forall w.A$, occurring in $\mathcal{T}$, has $|w| \leq 1$

Models, Subsumption, and Least Common Subsumer

- An interpretation $\mathcal{I}$ **satisfies** a GCI $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- $\mathcal{I}$ is a ***model* of** $\mathcal{T}$ iff it satisfies all GCIs in $\mathcal{T}$.
- C is **subsumed** by D w.r.t. $\mathcal{T}$ (denoted by $C \sqsubseteq_{\mathcal{T}} D$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{T}$. This relationship is called **subsumption**.

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- An $\mathcal{FL}_0$-concept E is the **least common subsumer**($\mathsf{lcs}_{\mathcal{T}}(C, D)$) of C and D w.r.t. $\mathcal{T}$ iff:
  - $C \sqsubseteq_{\mathcal{T}} E$ and $D \sqsubseteq_{\mathcal{T}} E$
  - For all concepts F such that $C \sqsubseteq_{\mathcal{T}} F$ and $D \sqsubseteq_{\mathcal{T}} F$, then $E \sqsubseteq_{\mathcal{T}} F$.

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Assumptions

In the following, w.l.o.g., we assume that the inputs are
A PANF TBox $\mathcal{T}$ and concept names $C, D \in N_{C,\mathcal{T}}$.

Functional Models of a Concept w.r.t. a TBox

- $\mathcal{I}$ is a **functional model of a concept C w.r.t. a TBox $\mathcal{T}$** iff
  - Complete n-ary tree, where $n = |N_R|$ (*tree-structured*);
  - For all r in N_R, $(u, v) \in r^{\mathcal{I}}$ iff $v = ur$ (*tree-structured*);
  - Satisfying all GCIs in $\mathcal{T}$ (*model of* $\mathcal{T}$);
  - Satisfying C at the root ($\varepsilon \in C^{\mathcal{I}}$).
- For all $w \in \Delta^{\mathcal{I}}$, the **label of w** in $\mathcal{I}$ is a set of concept names $A \in N_C$, where $w \in A^{\mathcal{I}}$.

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- Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be over the same domain elements.
  - **Subset relation between two functional models.**

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \text{ iff } A^{\mathcal{I}_1} \subseteq A^{\mathcal{I}_2} \text{ for all } A \in N_C$$

  - **Intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ between two functional models.**

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Functional Models of a Concept w.r.t. a TBox

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- Let $\mathcal{I}$ be a functional model of a TBox. **$(\mathcal{I}, u)$ is a subtree of $\mathcal{I}$** defined as follows:
  - It has the **same domain elements** as $\mathcal{I}$;
  - $A^{(\mathcal{I},u)} := \{w \in N_R^* \mid uw \in A^{\mathcal{I}}\}$, for all $A \in N_C$.

Least Functional Model

- $\mathcal{I}_{C,\mathcal{T}}$ is the **least functional model (LFM)** of a concept C w.r.t. a TBox $\mathcal{T}$ iff

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- **Example 3: TBox $\mathcal{T}_2$**

$$\begin{array}{lcl} \{Sw & \sqsubseteq & Ar \sqcap \forall c.Sg; \\ Pt & \sqsubseteq & Ar \sqcap \forall c.Pm; \\ Sg & \sqsubseteq & At \sqcap \forall m.Sw; \\ Pm & \sqsubseteq & At \sqcap \forall m.Pt; \\ Ar & \sqsubseteq & \forall c.At; \\ At & \sqsubseteq & \forall m.Ar\} \end{array}$$

Sw = *Songwriter*	**Ar** = *Artist*
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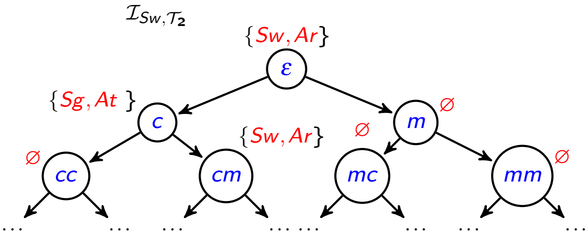
m = *madeUpBy*	**c** = *compose*



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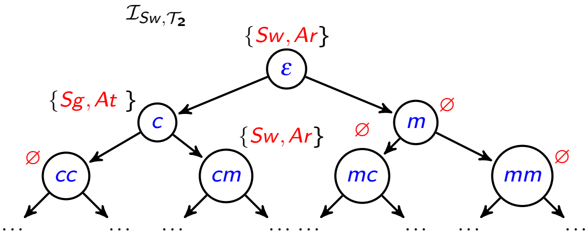

Why do we need LFMs? [Pensel, 2015]

$$C \sqsubseteq_{\mathcal{T}} D \text{ iff } \mathcal{I}_{D,\mathcal{T}} \subseteq \mathcal{I}_{C,\mathcal{T}}$$ (*Characterizing subsumption*)

Equivalence Class of Words

Labeling Function

For all $w \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$, we have a **labeling function**

$$\mathcal{I}_{C,\mathcal{T}}(w) := \{A \in N_{C,\mathcal{T}} \mid w \in A^{\mathcal{I}_{C,\mathcal{T}}}\}$$

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Let $u, v \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$. An **equivalence relation** $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ on $\Delta^{\mathcal{I}_{C,\mathcal{T}}}$ is defined as:

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Let $u \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$. The **equivalence class of words** u is defined as follows:
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Convention: Sometimes, to simplify the notation, we may omit $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ in $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$.

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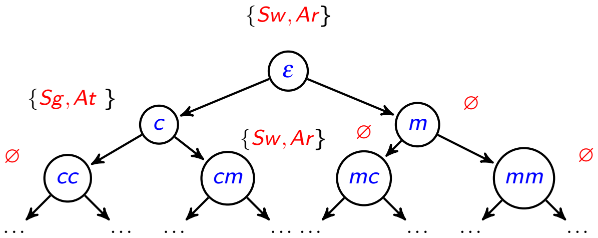
- The LFMs still have **infinite number** of elements with the same label.
- We construct the LFMs that only have a **finite number of elements** and ...
- ... change the form into a **cyclic fashion** → *graph of functional model*.

Graph of Least Functional Model

Example 4:

1. We have $\mathcal{I}_{Sw,\mathcal{T}_2}$

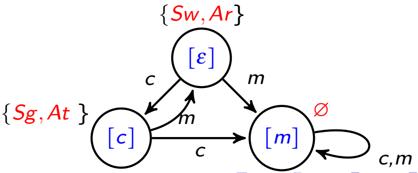


2. Equivalence class of words:
   - $[\varepsilon] = \{\varepsilon, cm \ldots\}$
   $\forall w \in [\varepsilon]$, $\mathcal{I}(w) = \{Sw, Ar\}$;

   - $[c] = \{c, cmc, \ldots\}$
   $\forall w \in [c]$, $\mathcal{I}(w) = \{Sg, At\}$;

   - $[m] = \{m, cc, mc, \ldots\}$
   $\forall w \in [m]$, $\mathcal{I}(w) = \varnothing$.

3. Construct the graph model $\mathcal{J}_{Sw,\mathcal{T}_2}$ (computing quotient structure $\Delta^{\mathcal{I}_{Sw,\mathcal{T}_2}} / \sim_{\mathcal{I}_{Sw,\mathcal{T}_2}}$)



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- $\mathcal{J}_{C,\mathcal{T}}$ is **effectively computable in a finite time.**
  - $\Delta^{\mathcal{J}_{C,\mathcal{T}}} \subseteq 2^{N_{C,\mathcal{T}}}$
    (*subsets of concept names occurring in $\mathcal{T}$ are finite*)
  - Initially, we have $[\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ with $\mathcal{I}(\varepsilon) = \{B \in N_{C,\mathcal{T}} \mid C \sqsubseteq_{\mathcal{T}} B\}$
    (*It is computable to find a maximal set from $N_{C,\mathcal{T}}$ s.t. all elements of the set subsume C w.r.t. $\mathcal{T}$*)
  - For each $r \in N_R$, we have

    $$([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}} \text{ iff for all } B \in \mathcal{I}(v), \text{ it holds } \sqcap \mathcal{I}(u) \sqsubseteq_{\mathcal{T}} \forall r.B$$

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Graph of Intersection Models

- Let $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ be the **graph models of $\mathcal{I}_{C,\mathcal{T}}$ and $\mathcal{I}_{D,\mathcal{T}}$**;
- Compute **the product $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$** of $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$;
- We take **a subgraph $\mathcal{G}$** of $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$, where all elements of $\mathcal{G}$ are reachable from $([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [\varepsilon]^{\sim_{\mathcal{I}_{D,\mathcal{T}}}})$
- $\mathcal{G}$ is **the graph model of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$**;

Simulation between Interpretations

- Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations.
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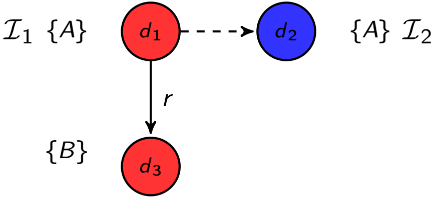

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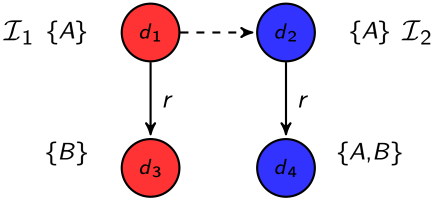

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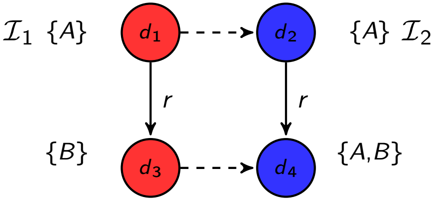

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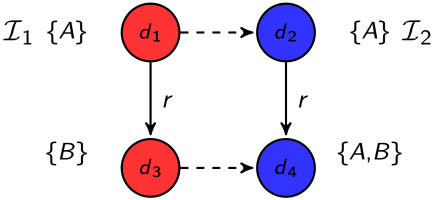

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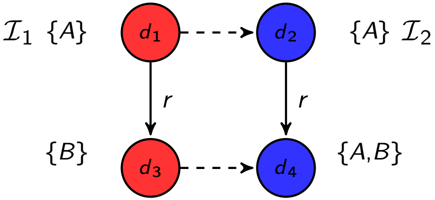

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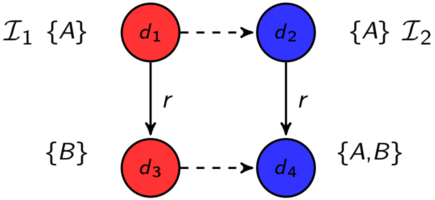

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Why do we need a simulation?

$C \sqsubseteq_{\mathcal{T}} D$ iff $\mathcal{J}_{D,\mathcal{T}} \lesssim \mathcal{J}_{C,\mathcal{T}}$ (*Characterizing subsumption*)

RP I: Is a Concept the LCS of C and D w.r.t. $\mathcal{T}$

A Condition whether a Concept is the LCS

Let E be an $\mathcal{FL}_0$-concept.

$$E \text{ is the } lcs_{\mathcal{T}}(C, D) \text{ iff } \mathcal{I}_{E,\mathcal{T}} = \mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$$

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$\Longrightarrow$ **RP I is decidable!**

$\mathcal{FL}_0$-Characteristic Concept

- The **role-depth** of a concept C ($rd(C)$) is the maximum number of $\forall$-quantifier in C.
- A **characteristic concept** K with $rd(K) = k$ can be obtained from a functional or graph model by traversing them until the depth k.

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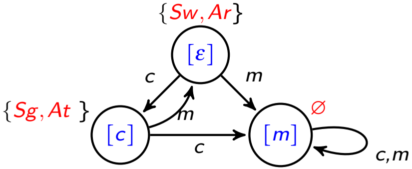

  - 0-characteristic concept of $\mathcal{J}_{Sw,\mathcal{T}} = Sw \sqcap Ar$;
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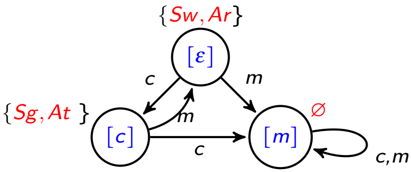

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- **Convention:** X^k is the k-characteristic concept of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ or $\mathcal{G}$, for $k \in \mathbb{N}$.

Label-Synchronous Elements

- Let $w \in \Delta^{\mathcal{I}_{\mathcal{C},\mathcal{T}}}$ and $Q = \bigsqcap\{B \in N_{\mathcal{C},\mathcal{T}} \mid B \in \mathcal{I}_{\mathcal{C},\mathcal{T}}(w)\}$.
  - $w \in \Delta^{\mathcal{I}_{\mathcal{C},\mathcal{T}}}$ is **label-synchronous in $\mathcal{I}_{\mathcal{C},\mathcal{T}}$** iff $(\mathcal{I}_{\mathcal{C},\mathcal{T}}, w) = (\mathcal{I}_{Q,\mathcal{T}}, \varepsilon)$
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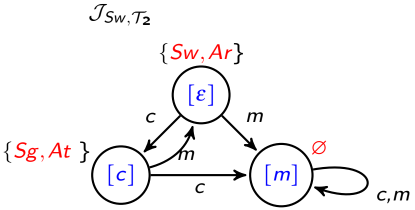

$[c]$ is label-synchronous in $\mathcal{J}_{Sw,\mathcal{T}_2}$ because $(\mathcal{J}_{Sw,\mathcal{T}_2}, [c]) \simeq (\mathcal{J}_{Sg \sqcap At,\mathcal{T}_2}, [\varepsilon])$

RP II: Does the LCS of C and D w.r.t. $\mathcal{T}$ exist?

Conditions for the Existence of the LCS

The $lcs_{\mathcal{T}}(C, D)$ exists iff there is a $k \in \mathbb{N}$ s.t.

- $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}$ iff
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Relationship between the LFM of X^k and Label-Synchronous Elements

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The $lcs_{\mathcal{T}}(C, D)$ exists iff all cycles in $\mathcal{G}$ only contains label-synchronous elements.

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$\implies$ **RP II is decidable!**

- $\mathcal{G}$ is computable in a finite time;
- Finitely many cycles in $\mathcal{G}$;
- It is decidable whether $[w]$ is label-synchronous in $\mathcal{G}$.

RP III: If the LCS exists, what is the LCS?

How to compute the LCS? And What is the Size of the LCS?

Let $n = \left|\Delta^{\mathcal{G}}\right|$. It holds that

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$\Longrightarrow$ **RP III is computable!**

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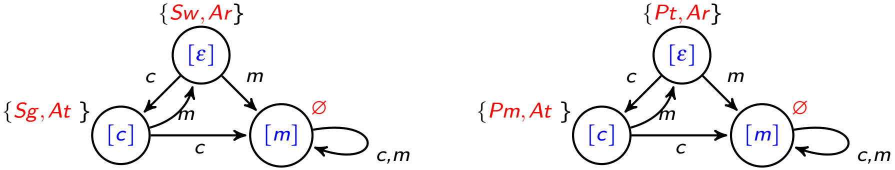

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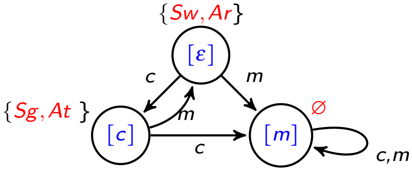

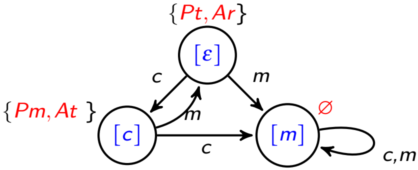

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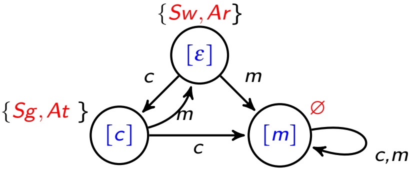

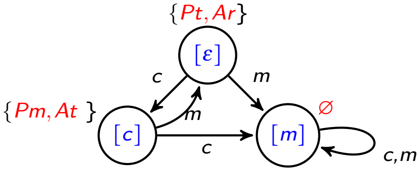

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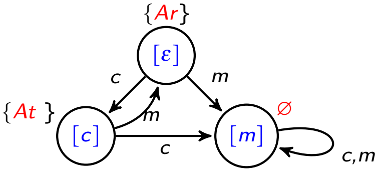

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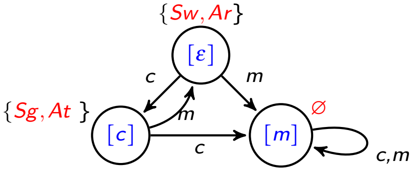
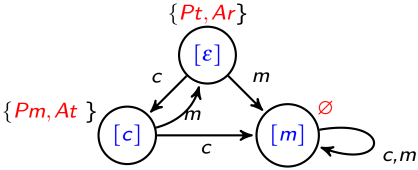
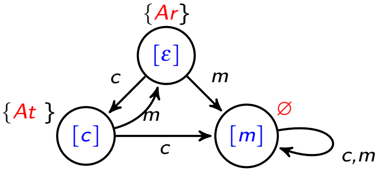
2. Compute $\mathcal{J}_{Sw,\mathcal{T}_2}$ and $\mathcal{J}_{Pt,\mathcal{T}_2}$;





3. Compute the product $\mathcal{J}_{Sw,\mathcal{T}_2} \times \mathcal{J}_{Pt,\mathcal{T}_2}$ of $\mathcal{J}_{Sw,\mathcal{T}_2}$ and $\mathcal{J}_{Pt,\mathcal{T}_2}$;
4. Compute the subgraph $\mathcal{G}$ of $\mathcal{J}_{Sw,\mathcal{T}_2} \times \mathcal{J}_{Pt,\mathcal{T}_2}$;



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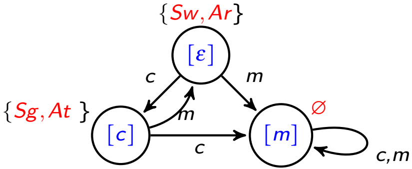

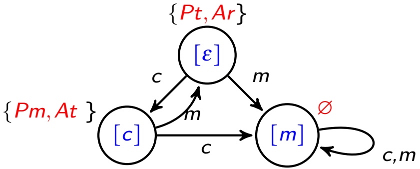

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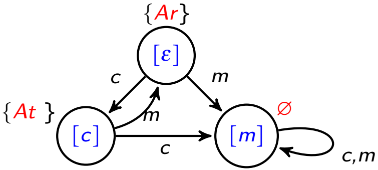

5. Since $|\Delta^{\mathcal{G}}| = 3$, we compute the 4-characteristic concept X^4 of $\mathcal{G}$ and construct $\mathcal{J}_{X^4,\mathcal{T}_2}$;
6. Check whether $(\mathcal{G}, [\varepsilon]) \simeq (\mathcal{J}_{X^4,\mathcal{T}}, [\varepsilon])$. Yes, X^4 **is the** $lcs_{\mathcal{T}_2}(Sw, Pt)$**!**
   Otherwise, the $lcs_{\mathcal{T}_2}(Sw, Pt)$ does not exist.

Conclusions and Future Works

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Future Works

- Practical implementation for the results above;
- Computing the lcs w.r.t. general $\mathcal{FLE}$-TBox;
- Computing the most specific concept of an individual w.r.t. general $\mathcal{FL}_0$-TBox.

Thank You