Algorithms for Computing Least Common Subsumers in General $\mathcal{FL}_0$-TBoxes

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Master’s Thesis

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September 27, 2016
1. Supporting knowledge engineers to **construct ontology by bottom-up approach**
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2. [Zarriess and Turhan, 2013] have found:
   - **A decision procedure** to decide the existence of the lcs w.r.t. general $\mathcal{EL}$-TBoxes
   - **An algorithm** for computing least common subsumers in general $\mathcal{EL}$-TBoxes
1. Supporting knowledge engineers to **construct ontology by bottom-up approach**

![Diagram](image)

2. [Zarriess and Turhan, 2013] have found:
   - A **decision procedure** to decide the existence of the lcs w.r.t. general $\mathcal{EL}$-TBoxes
   - An **algorithm** for computing least common subsumers in general $\mathcal{EL}$-TBoxes

3. How about $\mathcal{FL}_0$?
   - **No decision procedures** for the problem of the existence of the lcs w.r.t. general $\mathcal{FL}_0$-TBoxes.
   - **No algorithms** for computing least common subsumers in general $\mathcal{FL}_0$-TBoxes.
1. **Example 1: The lcs does not exist**

   TBox $T_1 :=$

   \[
   \begin{align*}
   \{ & \text{Songwriter } \sqsubseteq \text{Artist} \sqcap \forall \text{composes}.\text{Song} \\
   & \text{Poet } \sqsubseteq \text{Artist} \sqcap \forall \text{composes}.\text{Poem} \\
   & \text{Song } \sqsubseteq \text{Art} \sqcap \forall \text{madeUpBy}.\text{Songwriter} \\
   & \text{Poem } \sqsubseteq \text{Art} \sqcap \forall \text{madeUpBy}.\text{Poet}\}
   \end{align*}
   \]

   - The lcs of Songwriter and Poet w.r.t. $T_1$ does not exist.
   - Their cyclic definitions allow us to always find a more specific common subsumer of them.
   - Common subsumers of Songwriter and Poet w.r.t. $T_1$:
     1. Artist;
     2. Artist $\sqcap \forall$ composes.Art;
     3. Artist $\sqcap \forall$ madeUpBy.Songwriter;

2. **Example 2: The lcs exists**

   \[
   T_2 := T_1 \cup \{ \text{Artist} \sqcap \forall \text{composes}.\text{Art} \sqcap \forall \text{madeUpBy}.\text{Artist} \} \]

   The lcs of Songwriter and Poet w.r.t. $T_2$ is Artist.
A Poet Composes A Poem

1. **Example 1: The lcs does not exist**

TBox $\mathcal{T}_1 :=$

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- Their cyclic definitions allow us to always find a more specific common subsumer of them.
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  1. Artist;
  2. Artist $\sqcap \forall \text{composes}.\text{Art}$;
  3. Artist $\sqcap \forall \text{composes}.(\text{Art} \sqcap \forall \text{madeUpBy}.\text{Artist})$;
     ...

...
A Poet Composes A Poem

1. **Example 1: The lcs does not exist**
   
   \[ \text{TBox } T_1 := \]
   
   \[
   \{ \text{Songwriter} \subseteq \text{Artist} \land \forall \text{composes.Song} \\
   \text{Poet} \subseteq \text{Artist} \land \forall \text{composes.Poem} \\
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   - The lcs of Songwriter and Poet w.r.t. \( T_1 \) does not exist.
   - Their cyclic definitions allow us to always find a more specific common subsumer of them.
   - Common subsumers of Songwriter and Poet w.r.t. \( T_1 \):
     
     1. Artist;
     2. Artist \( \land \forall \text{composes.Art} \);
     3. Artist \( \land \forall \text{composes.(Art} \land \forall \text{madeUpBy.Artist)} \);
     
     ...

2. **Example 2: The lcs exists**

   \[ T_2 := T_1 \cup \]
   
   \[
   \{ \text{Artist} \subseteq \forall \text{composes.Art} \\
   \text{Art} \subseteq \forall \text{madeUpBy.Artist} \}
   \]

   The lcs of Songwriter and Poet w.r.t. \( T_2 \) is Artist.
Let $C, D, E$ be $\mathcal{FL}_0$-concepts and $T$ be a general $\mathcal{FL}_0$-TBox.

1. **Research Problem I (RP I):**
   
   Is concept $E$ the lcs of $C$ and $D$ w.r.t. $T$?

2. **Research Problem II (RP II):**
   
   Does the lcs of $C$ and $D$ w.r.t. $T$ exist?

3. **Research Problem III (RP III):**
   
   If the lcs of $C$ and $D$ w.r.t. $T$ exists, then what is the lcs? And how big is the size of the lcs?
Description Logic $\mathcal{FL}_0$

- $N_C$: set of concept names with $A \in N_C \rightarrow$ Songwriter, Poet, Song, Poem, . . .
- $N_R$: set of role names with $r \in N_R \rightarrow$ writes, composes, madeUpBy, arranges, . . .
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- $\mathcal{FL}_0$ concepts are built by using the following structures:

$$C, D ::= \top | A | C \sqcap D | \forall r. C$$
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- $\mathcal{FL}_0$ concepts are built by using the following structures:

$$C,D ::= \top \mid A \mid C \cap D \mid \forall r.C$$

- An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of:
  - $\Delta^\mathcal{I}$: a non-empty domain.
    Here we define $\Delta^\mathcal{I} = N_R^*$
  - $\cdot^\mathcal{I}$ with $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$
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- The mapping $\cdot^\mathcal{I}$ is extended to $\mathcal{FL}_0$-concepts

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Conventions:
- $\forall r_1.\forall r_2\ldots\forall r_n A \equiv \forall w. A$, where $w = r_1r_2\ldots r_n \in N^*_R$.
- $A \equiv \forall \epsilon. A$
A (general) $\mathcal{FL}_0$ TBox $\mathcal{T}$ is a finite set of General Concept Inclusions (GCIs) of the form of $C \sqsubseteq D$.

$N_{C,\mathcal{T}}$: set of concept names occurring in $\mathcal{T}$. 

Normalization in $\mathcal{FL}_0$-TBoxes \cite{Pensel2015} A concept is in concept-conjunction-normal-form (CCNF) iff it is of the form $\forall w_1. A_1 \sqsubseteq \ldots \sqsubseteq \forall w_n. A_n$, where $A_i \in N_C$ and $w_i \in N^*_R$, for all $1 \leq i \leq n$.

An $\mathcal{FL}_0$-TBox $\mathcal{T}$ is in plane-axiom-normal-form (PANF) iff all left- and right-hand sides of all GCIs in $\mathcal{T}$ are in CCNF; every $\forall w. A$, occurring in $\mathcal{T}$, has $\text{divides}(w) \leq 1$. 


A (general) $\mathcal{FL}_0$ TBox $T$ is a finite set of General Concept Inclusions (GCIs) of the form of $C \subseteq D$.

$N_C, T$: set of concept names occurring in $T$.

**Normalization in $\mathcal{FL}_0$-TBoxes [Pensel, 2015]**

A concept is in **concept-conjunction-normal-form (CCNF)** iff it is of the form
\[
\forall w_1.A_1 \cap \ldots \cap \forall w_n.A_n,
\]
where $A_i \in N_C$ and $w_i \in N^*_R$, for all $1 \leq i \leq n$.

An $\mathcal{FL}_0$-TBox $T$ is in **plane-axiom-normal-form (PANF)** iff
- All left- and right-hand sides of all GCIs in $T$ are in CCNF;
- Every $\forall w.A$, occurring in $T$, has $|w| \leq 1$.
An interpretation $\mathcal{I}$ satisfies a GCI $C \subseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$.

$\mathcal{I}$ is a model of $\mathcal{T}$ iff it satisfies all GCIs in $\mathcal{T}$.

$C$ is subsumed by $D$ w.r.t. $\mathcal{T}$ (denoted by $C \sqsubseteq_T D$) iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{T}$. This relationship is called subsumption.
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An $\mathcal{FL}_0$-concept $E$ is the least common subsumer ($\text{lcs}_\mathcal{T}(C, D)$) of $C$ and $D$ w.r.t. $\mathcal{T}$ iff:
- $C \sqsubseteq_\mathcal{T} E$ and $D \sqsubseteq_\mathcal{T} E$
- For all concepts $F$ such that $C \sqsubseteq_\mathcal{T} F$ and $D \sqsubseteq_\mathcal{T} F$, then $E \sqsubseteq_\mathcal{T} F$. 

Assumptions In the following, w.l.o.g., we assume that the inputs are a $\mathbb{PAN}$ TBox $\mathcal{T}$ and concept names $C, D \in \mathbb{N}_C, \mathbb{T}$. 

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An interpretation $\mathcal{I}$ satisfies a GCI $C \subseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$.

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An $\mathcal{FL}_0$-concept $E$ is the least common subsumer ($\text{lcs}_\mathcal{T}(C, D)$) of $C$ and $D$ w.r.t. $\mathcal{T}$ iff:
- $C \sqsubseteq_\mathcal{T} E$ and $D \sqsubseteq_\mathcal{T} E$
- For all concepts $F$ such that $C \sqsubseteq_\mathcal{T} F$ and $D \sqsubseteq_\mathcal{T} F$, then $E \sqsubseteq_\mathcal{T} F$.

**Assumptions**

In the following, w.l.o.g., we assume that the inputs are a PANF TBox $\mathcal{T}$ and concept names $C, D \in N_{C, \mathcal{T}}$. 
\( I \) is a **functional model of a concept** \( C \) w.r.t. a TBox \( T \) iff

- Complete \( n \)-ary tree, where \( n = |N_R| \) (tree-structured);
- For all \( r \in N_R \), \((u, v) \in r^I \) iff \( v = ur \) (tree-structured);
- Satisfying all GCIs in \( T \) (model of \( T \));
- Satisfying \( C \) at the root (\( \varepsilon \in C^I \)).

For all \( w \in \Delta^I \), the **label of** \( w \) in \( I \) is a set of concept names \( A \in N_C \), where \( w \in A^I \).
$\mathcal{I}$ is a **functional model of a concept $C$ w.r.t. a TBox $\mathcal{T}$** iff

- Complete $n$-ary tree, where $n = |N_R|$ (*tree-structured*);
- For all $r$ in $N_R$, $(u,v) \in r^\mathcal{I}$ iff $v = ur$ (*tree-structured*);
- Satisfying all GCIs in $\mathcal{T}$ (*model of $\mathcal{T}$*);
- Satisfying $C$ at the root ($\varepsilon \in C^\mathcal{I}$).

For all $w \in \Delta^\mathcal{I}$, the **label of $w$** in $\mathcal{I}$ is a set of concept names $A \in N_C$, where $w \in A^\mathcal{I}$.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be over the same domain elements.

- **Subset relation between two functional models.**
  $$\mathcal{I}_1 \subseteq \mathcal{I}_2 \text{ iff } A^{\mathcal{I}_1} \subseteq A^{\mathcal{I}_2} \text{ for all } A \in N_C$$

- **Intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ between two functional models.**
  $$A^{\mathcal{I}_1 \cap \mathcal{I}_2} \text{ iff } A^{\mathcal{I}_1} \cap A^{\mathcal{I}_2} \text{ for all } A \in N_C$$
\[ I \] is a **functional model of a concept** \( C \) w.r.t. a TBox \( \mathcal{T} \) iff

- Complete \( n \)-ary tree, where \( n = |N_R| \) (tree-structured);
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- Satisfying \( C \) at the root (\( \epsilon \in C^I \)).

For all \( w \in \Delta^I \), the **label** of \( w \) in \( I \) is a set of concept names \( A \in N_C \), where \( w \in A^I \).

Let \( I_1 \) and \( I_2 \) be over the same domain elements.

- **Subset relation between two functional models.**
  \[ I_1 \subseteq I_2 \] iff \( A^{I_1} \subseteq A^{I_2} \) for all \( A \in N_C \)

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  \[ A^{I_1} \cap I_2 \] iff \( A^{I_1} \cap A^{I_2} \) for all \( A \in N_C \)

Let \( I \) be a functional model of a TBox. \((I, u)\) is a **subtree** of \( I \) defined as follows:

- It has the **same domain elements** as \( I \);
- \( A^{(I,u)} := \{ w \in N^*_R \mid uw \in A^I \} \), for all \( A \in N_C \).
Least Functional Model

- $\mathcal{I}_{C,T}$ is the **least functional model (LFM)** of a concept $C$ w.r.t. a TBox $T$ iff $\mathcal{I}_{C,T} \subseteq \mathcal{I}$ for all functional models $\mathcal{I}$ of $C$ w.r.t. $T$.

Example 3: TBox $T_2$:

- $\text{Sw} \sqsubseteq \text{Ar}$
- $\forall c. \text{Sg} \sqsubseteq \text{Pt}$
- $\forall m. \text{Sw} \sqsubseteq \text{Pm}$
- $\forall c. \text{At} \sqsubseteq \text{Ar}$
- $\forall m. \text{Pt} \sqsubseteq \text{Ar}$

$\text{Sw} = \text{Songwriter}, \text{Ar} = \text{Artist}, \text{Sg} = \text{Song}, \text{At} = \text{Art}, \text{Pt} = \text{Poet}, \text{Pm} = \text{Poem}$

$\forall c. \text{compose} \in \{\text{Sw}, \text{Ar}\}$

Why do we need LFMs? [Pensel, 2015]
Least Functional Model

- $\mathcal{I}_{C,T}$ is the **least functional model (LFM)** of a concept $C$ w.r.t. a TBox $\mathcal{T}$ iff $\mathcal{I}_{C,T} \subseteq \mathcal{I}$ for all functional models $\mathcal{I}$ of $C$ w.r.t. $\mathcal{T}$.

**Example 3: TBox $\mathcal{T}_2$**

\[
\begin{align*}
\{S\text{w} & \subseteq \text{A}r \cap \forall c.\text{Sg}; \\
\text{Pt} & \subseteq \text{A}r \cap \forall c.\text{Pm}; \\
\text{Sg} & \subseteq \text{A}t \cap \forall m.\text{Sw}; \\
\text{Pm} & \subseteq \text{A}t \cap \forall m.\text{Pt}; \\
\text{Ar} & \subseteq \forall c.\text{At}; \\
\text{At} & \subseteq \forall m.\text{Ar}\}
\end{align*}
\]

Sw = Songwriter  \quad Ar = Artist  \quad Sg = Song  \quad At = Art  \quad Pt = Poet  \quad Pm = Poem  \quad m = madeUpBy  \quad c = compose
Least Functional Model

- $I_{C,T}$ is the **least functional model (LFM)** of a concept $C$ w.r.t. a TBox $T$ iff $I_{C,T} \subseteq I$ for all functional models $I$ of $C$ w.r.t. $T$.

**Example 3: TBox $T_2$**

$\{Sw \subseteq Ar \cap \forall c.Sg; 
Pt \subseteq Ar \cap \forall c.Pm; 
Sg \subseteq At \cap \forall m.Sw; 
Pm \subseteq At \cap \forall m.Pt; 
Ar \subseteq \forall c.At; 
At \subseteq \forall m.Ar\}$

$Sw = Songwriter \quad Ar = Artist 
Sg = Song \quad At = Art 
Pt = Poet \quad Pm = Poem 
m = madeUpBy \quad c = compose$

Why do we need LFM$s$? [Pensel, 2015]

$C \sqsubseteq_T D$ iff $I_{D,T} \subseteq I_{C,T}$ (Characterizing subsumption)
Equivalence Class of Words

Labeling Function

For all $w \in \Delta^{\mathcal{I}_C,\mathcal{T}}$, we have a labeling function

$$\mathcal{I}_{C,T}(w) := \{ A \in N_{C,T} \mid w \in A^{\mathcal{I}_C,\mathcal{T}} \}$$
Equivalence Class of Words

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For all $w \in \Delta^{I_C,T}$, we have a **labeling function**

$$I_{C,T}(w) := \{ A \in N_{C,T} \mid w \in A^{I_C,T} \}$$

Equivalent Class of Words

Let $u, v \in \Delta^{I_C,T}$. An **equivalence relation** $\sim_{I_C,T}$ on $\Delta^{I_C,T}$ is defined as:

$$u \sim_{I_C,T} v \text{ iff } I_{C,T}(u) = I_{C,T}(v)$$
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For all $w \in \Delta^{I_C, T}$, we have a **labeling function**

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Equivalence Class of Words

Let $u \in \Delta^{I_C, T}$. The **equivalence class of words** $u$ is defined as follows:

$$[u]^{I_C, T} := \{ v \in \Delta^{I_C, T} \mid u \sim^{I_C, T} v \}$$

**Convention:** Sometimes, to simplify the notation, we may omit $\sim^{I_C, T}$ in $[u]^{I_C, T}$. 

The LFMs still have infinite number of elements with the same label. We construct the LFMs that only have a finite number of elements and change the form into a cyclic fashion to graph of functional model.
### Equivalence Class of Words

#### Labeling Function

For all \( w \in \Delta^I \), we have a **labeling function**

\[
I_C,T(w) := \{ A \in N_C,T \mid w \in A^I \}
\]

#### Equivalence Relation

Let \( u, v \in \Delta^I \). An **equivalence relation** \( \sim_{I_C,T} \) on \( \Delta^I \) is defined as:

\[
u \sim_{I_C,T} v \text{ iff } I_C,T(u) = I_C,T(v)\]

#### Equivalence Class of Words

Let \( u \in \Delta^I \). The **equivalence class of words** \( u \) is defined as follows:

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[u]_{I_C,T} := \{ v \in \Delta^I \mid u \sim_{I_C,T} v \}
\]

**Convention:** Sometimes, to simplify the notation, we may omit \( \sim_{I_C,T} \) in \([u]_{I_C,T}\).

- The LFM still have **infinite number** of elements with the same label.
- We construct the LFM that only have a **finite number of elements** and . . .
- . . . change the form into a **cyclic fashion** → **graph of functional model**.
Example 4:

1. We have $\mathcal{I}_{Sw,T_2}$

   \[ \begin{array}{c}
   \{ Sw, Ar \} \\
   \{ Sg, At \} \\
   \{ c, cmc, ... \} \\
   \{ m, cc, mc, ... \} \\
   \end{array} \]

2. Equivalence class of words:
   - $[\varepsilon] = \{ \varepsilon, cm, ... \}$
     \[ \forall w \in [\varepsilon], \mathcal{I}(w) = \{ Sw, Ar \}; \]
   - $[c] = \{ c, cmc, ... \}$
     \[ \forall w \in [c], \mathcal{I}(w) = \{ Sg, At \}; \]
   - $[m] = \{ m, cc, mc, ... \}$
     \[ \forall w \in [m], \mathcal{I}(w) = \emptyset. \]

3. Construct the graph model $\mathcal{J}_{Sw,T_2}$ (computing quotient structure $\Delta^{\mathcal{I}_{Sw,T_2}} / \sim^{\mathcal{I}_{Sw,T_2}}$)
\( J_{C,T} \) is **effectively computable in a finite time.**

- \( \Delta^{J_{C,T}} \subseteq 2^{N_{C,T}} \)
  (subsets of concept names occurring in \( T \) are finite)
- Initially, we have \( [\varepsilon]^\sim_{I_{C,T}} \) with \( I(\varepsilon) = \{ B \in N_{C,T} \mid C \subseteq_T B \} \)
  (It is computable to find a maximal set from \( N_{C,T} \) s.t. all elements of the set subsume \( C \) w.r.t. \( T \))
- For each \( r \in N_{R} \), we have
  \[
  ([u]^\sim_{I_{C,T}}, [v]^\sim_{I_{C,T}}) \in r^{J_{C,T}} \text{ iff for all } B \in I(v), \text{ it holds } \sqcap I(u) \subseteq_T \forall r.B
  \]
  (It is computable to find a maximal set from \( N_{C,T} \) s.t. for all elements \( B \) of the set, we have \( \forall r.B \) subsumes \( \sqcap I(u) \) w.r.t. \( T \))
Graph of Least Functional Model

- $\mathcal{J}_{C,T}$ is effectively computable in a finite time.
  - $\Delta^{\mathcal{J}_{C,T}} \subseteq 2^{\mathcal{N}_{C,T}}$
    (subsets of concept names occurring in $\mathcal{T}$ are finite)
  - Initially, we have $[\varepsilon]^{\mathcal{I}_{C,T}}$ with $\mathcal{I}(\varepsilon) = \{ B \in \mathcal{N}_{C,T} \mid C \sqsubseteq_T B \}$
    (It is computable to find a maximal set from $\mathcal{N}_{C,T}$ s.t. all elements of the set subsume $C$ w.r.t. $\mathcal{T}$)
  - For each $r \in \mathcal{N}_R$, we have $([u]^{\mathcal{I}_{C,T}}, [v]^{\mathcal{I}_{C,T}}) \in r^{\mathcal{J}_{C,T}}$ iff for all $B \in \mathcal{I}(v)$, it holds $\cap \mathcal{I}(u) \sqsubseteq_T \forall r.B$
    (It is computable to find a maximal set from $\mathcal{N}_{C,T}$ s.t. for all elements $B$ of the set, we have $\forall r.B$ subsumes $\cap \mathcal{I}(u)$ w.r.t. $\mathcal{T}$)

Graph of Intersection Models

- Let $\mathcal{J}_{C,T}$ and $\mathcal{J}_{D,T}$ be the graph models of $\mathcal{I}_{C,T}$ and $\mathcal{I}_{D,T}$;
- Compute the product $\mathcal{J}_{C,T} \times \mathcal{J}_{D,T}$ of $\mathcal{J}_{C,T}$ and $\mathcal{J}_{D,T}$;
- We take a subgraph $\mathcal{G}$ of $\mathcal{J}_{C,T} \times \mathcal{J}_{D,T}$, where all elements of $\mathcal{G}$ are reachable from $([\varepsilon]^{\mathcal{I}_{C,T}}, [\varepsilon]^{\mathcal{I}_{D,T}})$
- $\mathcal{G}$ is the graph model of $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$;
Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is defined as a simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$. 

Why do we need a simulation?
Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is defined as a simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.

**Example 5:**

\[
\begin{array}{c}
\mathcal{I}_1 \{A\} \xrightarrow{d_1} \{A\} \mathcal{I}_2 \quad ((\mathcal{I}_1, d_1) \text{ is simulated } (\preceq) \text{ by } (\mathcal{I}_2, d_2)) \\
\{B\} \xrightarrow{r} \{B\}
\end{array}
\]
Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is defined as a simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.

**Example 5:**

\[ \mathcal{I}_1 \{A\} \rightarrow d_1 \rightarrow d_2 \rightarrow \mathcal{I}_2 \{A\} \]

\[ ((\mathcal{I}_1, d_1) \text{ is simulated } (\subseteq) \text{ by } (\mathcal{I}_2, d_2)) \]

\[ \mathcal{I}_1 \{B\} \rightarrow d_3 \rightarrow d_4 \rightarrow \mathcal{I}_2 \{A,B\} \]
Let \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) be interpretations.

\[ S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2} \]

is defined as a simulation from \( \mathcal{I}_1 \) to \( \mathcal{I}_2 \).

**Example 5:**

\[
\begin{align*}
\mathcal{I}_1 & \{A\} \quad \downarrow \quad \{A\} \quad \mathcal{I}_2 \\
\{B\} & \quad \downarrow \quad \{A, B\}
\end{align*}
\]

\((\mathcal{I}_1, d_1)\) is simulated \((\subseteq)\) by \((\mathcal{I}_2, d_2)\)

\((\mathcal{I}_1, d_3)\) is simulated \((\subseteq)\) by \((\mathcal{I}_2, d_4)\)
Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is defined as a simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.

Example 5:

$\mathcal{I}_1 \{A\}$ $\xrightarrow{d_1} \mathcal{I}_2 \{A\}$ ($(\mathcal{I}_1, d_1)$ is simulated ($\preceq$) by $(\mathcal{I}_2, d_2)$)

$\mathcal{I}_1 \{A\}$ $\xrightarrow{d_3} \mathcal{I}_2 \{A,B\}$ ($(\mathcal{I}_1, d_3)$ is simulated ($\preceq$) by $(\mathcal{I}_2, d_4)$)

$(\mathcal{I}_1,d)$ is simulation-equivalent to $(\mathcal{I}_2,e)$ (denoted by $(\mathcal{I}_1,d) \simeq (\mathcal{I}_2,e)$) if $(\mathcal{I}_1,d) \preceq (\mathcal{I}_2,e)$ and $(\mathcal{I}_2,e) \preceq (\mathcal{I}_1,d)$. 
Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is defined as a simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.

**Example 5:**

$(\mathcal{I}_1, d_1)$ is simulated ($\subseteq$) by $(\mathcal{I}_2, d_2)$

$(\mathcal{I}_1, d_3)$ is simulated ($\subseteq$) by $(\mathcal{I}_2, d_4)$

$(\mathcal{I}_1, d)$ is simulation-equivalent to $(\mathcal{I}_2, e)$ (denoted by $(\mathcal{I}_1, d) \simeq (\mathcal{I}_2, e)$) if $(\mathcal{I}_1, d) \subseteq (\mathcal{I}_2, e)$ and $(\mathcal{I}_2, e) \subseteq (\mathcal{I}_1, d)$.

This notion is applied analogously to functional models and graph models.
Simulation between Interpretations

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is defined as a simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.

**Example 5:**

\[
\begin{align*}
\mathcal{I}_1 & \{A\} \quad d_1 \quad d_2 \quad \{A\} \quad \mathcal{I}_2 \quad ((\mathcal{I}_1, d_1) \text{ is simulated } \preceq \text{ by } (\mathcal{I}_2, d_2)) \\
& \downarrow r \quad \quad \quad \downarrow r \\
\{B\} & \quad d_3 \quad d_4 \quad \{A,B\} \\
& ((\mathcal{I}_1, d_3) \text{ is simulated } \preceq \text{ by } (\mathcal{I}_2, d_4))
\end{align*}
\]

$(\mathcal{I}_1,d)$ is simulation-equivalent to $(\mathcal{I}_2,e)$ (denoted by $(\mathcal{I}_1,d) \simeq (\mathcal{I}_2,e)$) if $(\mathcal{I}_1,d) \preceq (\mathcal{I}_2,e)$ and $(\mathcal{I}_2,e) \preceq (\mathcal{I}_1,d)$.

This notion is applied analogously to functional models and graph models.

**Why do we need a simulation?**

$C \subseteq_{T} D$ iff $\mathcal{J}_{D,T} \preceq \mathcal{J}_{C,T}$ (Characterizing subsumption)
A Condition whether a Concept is the LCS

Let $E$ be an $\mathcal{FL}_0$-concept.

$E$ is the $\text{lcs}_T(C, D)$ iff $\mathcal{I}_{E,T} = \mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$
A Condition whether a Concept is the LCS

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$E$ is the $\text{lcs}_T(C, D)$ iff $\mathcal{I}_{E,T} = \mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$

$\implies \mathcal{I}_{E,T}$ and $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$ are infinite models!
A Condition whether a Concept is the LCS

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A Condition whether a Concept is the LCS

Let $E$ be an $\mathcal{FL}_0$-concept.

$E$ is the $lcs_T(C, D)$ iff $\mathcal{J}_{E,T} \sim \mathcal{G}$
A Condition whether a Concept is the LCS

Let \( E \) be an FL\(_0\)-concept.

\[ E \text{ is the } lcs_T(C, D) \text{ iff } I_{E,T} = I_{C,T} \cap I_{D,T} \]

\[ \implies I_{E,T} \text{ and } I_{C,T} \cap I_{D,T} \text{ are infinite models!} \]

A Condition whether a Concept is the LCS

Let \( E \) be an FL\(_0\)-concept.

\[ E \text{ is the } lcs_T(C, D) \text{ iff } J_{E,T} \simeq G \]

\[ \implies \text{RP I is decidable!} \]
The **role-depth** of a concept $C$ ($rd(C)$) is the maximum number of $\forall$-quantifier in $C$.

A **characteristic concept** $K$ with $rd(K) = k$ can be obtained from a functional or graph model by traversing them until the depth $k$. 

Convention: $X_k$ is the $k$-characteristic concept of $I_C$, $T \cap I_D$, $T$ or $G$, for $k \in \mathbb{N}$. 
The **role-depth** of a concept $C$ ($rd(C)$) is the maximum number of $\forall$-quantifier in $C$.

A **characteristic concept** $K$ with $rd(K) = k$ can be obtained from a functional or graph model by traversing them until the depth $k$.

**Example 6:** $\mathcal{J}_{Sw,T_2}$

- 0-characteristic concept of $\mathcal{J}_{Sw,T} = Sw \cap Ar$;
- 1-characteristic concept of $\mathcal{J}_{Sw,T} = Sw \cap Ar \cap \forall c.Sg \cap \forall c.At \cap \forall m.T$;
- 2-characteristic concept of $\mathcal{J}_{Sw,T} = Sw \cap Ar \cap \forall c.Sg \cap \forall c.At \cap \forall m.T \cap \forall cc.T \cap \forall cm.Sw \cap \forall cm.Ar \cap \forall cc.T \cap \forall cm.T$
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**Convention:** $X^k$ is the $k$-characteristic concept of $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$ or $\mathcal{G}$, for $k \in \mathbb{N}$. 
Label-Synchronous Elements

Let \( w \in \Delta^{\mathcal{I}_C, \mathcal{T}} \) and \( Q = \bigcap \{ B \in \mathcal{N}_{C, \mathcal{T}} \mid B \in \mathcal{I}_{C, \mathcal{T}}(w) \} \).

- \( w \in \Delta^{\mathcal{I}_C, \mathcal{T}} \) is **label-synchronous in** \( \mathcal{I}_{C, \mathcal{T}} \) iff \( (\mathcal{I}_{C, \mathcal{T}}, w) = (\mathcal{I}_Q, \mathcal{T}, e) \)
- \( \lfloor w \rfloor \) is **label-synchronous in** \( \mathcal{J}_{C, \mathcal{T}} \) iff \( (\mathcal{J}_{C, \mathcal{T}}, \lfloor w \rfloor) \simeq (\mathcal{J}_Q, \mathcal{T}, \lfloor e \rfloor) \)
Let \( w \in \Delta^{I_{C,T}} \) and \( Q = \bigcap \{ B \in N_{C,T} \mid B \in I_{C,T}(w) \} \).

- \( w \in \Delta^{I_{C,T}} \) is \textbf{label-synchronous in} \( I_{C,T} \) iff \( (I_{C,T}, w) = (I_Q,T,\varepsilon) \)
- \( [w] \) is \textbf{label-synchronous in} \( J_{C,T} \) iff \( (J_{C,T}, [w]) \simeq (J_Q,T,[\varepsilon]) \)

**Example 7:**

\( J_{Sw,T} \)

\[ \{Sw, Ar\} \]

\[ [\varepsilon] \]

\[ \{Sg, At\} \]

\[ [c] \]

\( c \)

\( m \)

\( m \)

\( \emptyset \)

\( c \)

\( c,m \)

\( c \)

\( c,m \)

\( [m] \)

\( [c] \) is label-synchronous in \( J_{Sw,T_2} \) because \( (J_{Sw,T_2}, [c]) \simeq (J_{Sg\cap At,T_2}, [\varepsilon]) \)
### Conditions for the Existence of the LCS

The $\text{lcs}_T(C, D)$ exists iff there is a $k \in \mathbb{N}$ s.t.

1. $\mathcal{I}_{C, T} \cap \mathcal{I}_{D, T} = \mathcal{I}_{X^k, T}$ iff
2. $(G, [\varepsilon]) \simeq (\mathcal{J}_{X^k, T}, [\varepsilon])$. 

RP II: Does the LCS of $C$ and $D$ w.r.t. $T$ exist?
RP II: Does the LCS of $C$ and $D$ w.r.t. $\mathcal{T}$ exist?

### Conditions for the Existence of the LCS

The $\text{lcs}_{\mathcal{T}}(C, D)$ exists iff there is a $k \in \mathbb{N}$ s.t.

- $\mathcal{I}_{C, \mathcal{T}} \cap \mathcal{I}_{D, \mathcal{T}} = \mathcal{I}_{X^k, \mathcal{T}}$ iff
- $(\mathcal{G}, [\varepsilon]) \simeq (\mathcal{J}_{X^k, \mathcal{T}}, [\varepsilon])$.

$\implies$ infinitely many $k$; $\mathcal{I}_{C, \mathcal{T}} \cap \mathcal{I}_{D, \mathcal{T}}$ and $\mathcal{I}_{X^k, \mathcal{T}}$ are infinite models.
RP II: Does the LCS of $C$ and $D$ w.r.t. $\mathcal{T}$ exist?

Conditions for the Existence of the LCS

The $lcs_{\mathcal{T}}(C, D)$ exists iff there is a $k \in \mathbb{N}$ s.t.

- $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}$ iff
- $(\mathcal{G}, [\mathcal{E}]) \simeq (\mathcal{I}_{X^k,\mathcal{T}}, [\mathcal{E}])$.

$\implies$ infinitely many $k$; $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{X^k,\mathcal{T}}$ are infinite models.

Relationship between the LFM of $X^k$ and Label-Synchronous Elements

$\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}$ iff for all $w \in \mathbb{N}_R^*$ with $|w| \geq k$, it holds that

- $w$ is label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{X^k,\mathcal{T}}$.
RP II: Does the LCS of $C$ and $D$ w.r.t. $\mathcal{T}$ exist?

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The $lcs_\mathcal{T}(C, D)$ exists iff there is a $k \in \mathbb{N}$ s.t.

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**Main Theorem**

The $lcs_\mathcal{T}(C, D)$ exists iff all cycles in $\mathcal{G}$ only contains label-synchronous elements.
RP II: Does the LCS of $C$ and $D$ w.r.t. $\mathcal{T}$ exist?

**Conditions for the Existence of the LCS**

The $lcs_{\mathcal{T}}(C, D)$ exists iff there is a $k \in \mathbb{N}$ s.t.

- $\mathcal{I}_{C, \mathcal{T}} \cap \mathcal{I}_{D, \mathcal{T}} = \mathcal{I}_{X^k, \mathcal{T}}$ iff
- $(\mathcal{G}, [\varepsilon]) \cong (\mathcal{J}_{X^k, \mathcal{T}}, [\varepsilon])$.

$\implies$ infinitely many $k$; $\mathcal{I}_{C, \mathcal{T}} \cap \mathcal{I}_{D, \mathcal{T}}$ and $\mathcal{I}_{X^k, \mathcal{T}}$ are infinite models.

**Relationship between the LFM of $X^k$ and Label-Synchronous Elements**

$\mathcal{I}_{C, \mathcal{T}} \cap \mathcal{I}_{D, \mathcal{T}} = \mathcal{I}_{X^k, \mathcal{T}}$ iff for all $w \in \mathbb{N}_R^*$ with $|w| \geq k$, it holds that

- $w$ is label-synchronous in $\mathcal{I}_{C, \mathcal{T}} \cap \mathcal{I}_{D, \mathcal{T}}$ and $\mathcal{I}_{X^k, \mathcal{T}}$.

**Main Theorem**

The $lcs_{\mathcal{T}}(C, D)$ exists iff all cycles in $\mathcal{G}$ only contains label-synchronous elements.

$\implies$ **RP II is decidable!**

- $\mathcal{G}$ is computable in a finite time;
- Finitely many cycles in $\mathcal{G}$;
- It is decidable whether $[w]$ is label-synchronous in $\mathcal{G}$.
RP III: If the LCS exists, what is the LCS?

How to compute the LCS? And What is the Size of the LCS?

Let \( n = |\Delta^G| \). It holds that

- The \( lcs_T(C, D) \) exists iff \( (G, [\varepsilon]) \simeq (J_{X^{n+1}}, T, [\varepsilon]) \);
  - \( X^{n+1} \) is the \( lcs_T(C, D) \).
- \( rd(lcs_T(C, D)) \leq 2^{2^{|N_C, T|+1}} \).
How to compute the LCS? And What is the Size of the LCS?

Let $n = |\Delta^G|$. It holds that

- The $lcs_T(C, D)$ exists iff $(G, [\varepsilon]) \simeq (J_{X_{n+1}}, T, [\varepsilon])$;
  - $X^{n+1}$ is the $lcs_T(C, D)$.
- $rd(lcs_T(C, D)) \leq 2^{2^{|N_{C,T}|+1}}$.

$\implies$ RP III is computable!
An Algorithm to Compute the LCS, if it Exists

1. Given $\mathcal{T}_2$ in PANF and $Sw, Pt \in N_{C, \mathcal{T}_2}$;
An Algorithm to Compute the LCS, if it Exists

1. Given $T_2$ in PANF and $Sw, Pt \in N_{C,T_2}$;
2. Compute $J_{Sw, T_2}$ and $J_{Pt, T_2}$;

3. Compute the product $J_{Sw, T_2} \times J_{Pt, T_2}$ of $J_{Sw, T_2}$ and $J_{Pt, T_2}$;
4. Compute the subgraph $G$ of $J_{Sw, T_2} \times J_{Pt, T_2}$;

5. Since $\Delta G = 3$, we compute the 4-characteristic concept $X_4$ of $G$ and construct $J_{X_4, T_2}$;
6. Check whether $(G, \varepsilon)$ isomorphic $(J_{X_4, T_2}, \varepsilon)$. Yes, $X_4$ is the lcs $T_2(Sw, Pt)$.
Otherwise, the lcs $T_2(Sw, Pt)$ does not exist.
An Algorithm to Compute the LCS, if it Exists

1. Given $T_2$ in PANF and $Sw, Pt \in N_{C,T_2}$;
2. Compute $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
3. Compute the product $J_{Sw,T_2} \times J_{Pt,T_2}$ of $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
4. Compute the subgraph $G$ of $J_{Sw,T_2} \times J_{Pt,T_2}$;
5. Since $\Delta G = 3$, we compute the 4-characteristic concept $X_4$ of $G$ and construct $J_{X_4,T_2}$;
6. Check whether $(G, \varepsilon) \equiv (J_{X_4,T_2}, \varepsilon)$. Yes, $X_4$ is the lcs $T_2(Sw, Pt)$! Otherwise, the lcs $T_2(Sw, Pt)$ does not exist.
An Algorithm to Compute the LCS, if it Exists

1. Given $T_2$ in PANF and $Sw, Pt \in N_{C,T_2}$;
2. Compute $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
3. Compute the product $J_{Sw,T_2} \times J_{Pt,T_2}$ of $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
4. Compute the subgraph $G$ of $J_{Sw,T_2} \times J_{Pt,T_2}$;
5. Since $G$ divides $\Delta = 3$, we compute the 4-characteristic concept $X_4$ of $G$ and construct $J_{X_4,T_2}$;
6. Check whether $(G, \varepsilon)$ divides $(J_{X_4,T_2}, \varepsilon)$. Yes, $X_4$ is the LCS of $(Sw, Pt)$! Otherwise, the LCS of $(Sw, Pt)$ does not exist.
An Algorithm to Compute the LCS, if it Exists

1. Given $T_2$ in PANF and $Sw, Pt \in N_{C,T_2}$;
2. Compute $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
3. Compute the product $J_{Sw,T_2} \times J_{Pt,T_2}$ of $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
4. Compute the subgraph $G$ of $J_{Sw,T_2} \times J_{Pt,T_2}$;
5. Since $|\Delta^G| = 3$, we compute the 4-characteristic concept $X^4$ of $G$ and construct $J_{X^4,T_2}$;
An Algorithm to Compute the LCS, if it Exists

1. Given $T_2$ in PANF and $Sw, Pt \in N_{C,T_2}$;
2. Compute $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
3. Compute the product $J_{Sw,T_2} \times J_{Pt,T_2}$ of $J_{Sw,T_2}$ and $J_{Pt,T_2}$;
4. Compute the subgraph $G$ of $J_{Sw,T_2} \times J_{Pt,T_2}$;
5. Since $|\Delta^G| = 3$, we compute the 4-characteristic concept $X^4$ of $G$ and construct $J_{X^4,T_2}$;
6. Check whether $(G, [\varepsilon]) \simeq (J_{X^4,T}, [\varepsilon])$. Yes, $X^4$ is the $lcs_{T_2}(Sw, Pt)$! Otherwise, the $lcs_{T_2}(Sw, Pt)$ does not exist.
Conclusions

**RP I:** An $\mathcal{FL}_0$-concept $E$ is the $lcs_T(C,D)$ iff

1. $I_{E,T} = I_{C,T} \cap I_{D,T}$;
2. $(J_{E,T},[\varepsilon]) \simeq (G,[\varepsilon])$.

Future Works

- Practical implementation for the results above;
- Computing the $lcs$ w.r.t. general $\mathcal{FL}_0$-TBox;
- Computing the most specific concept of an individual w.r.t. general $\mathcal{FL}_0$-TBox.
Conclusions and Future Works

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  - All cycles in $\mathcal{G}$ only contains label-synchronous elements.

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**RP I:** An $\mathcal{FL}_0$-concept $E$ is the $lcs_T(C, D)$ iff

- $I_{E,T} = I_{C,T} \cap I_{D,T}$;
- $(J_{E,T} , [\varepsilon]) \simeq (G , [\varepsilon])$.

**RP II:** The $lcs_T(C, D)$ exists iff

- There is a $k \in \mathbb{N}$ s.t. $I_{C,T} \cap I_{D,T} = I_{X_k,T}$;
- There is a $k \in \mathbb{N}$ s.t. $(G , [\varepsilon]) \simeq J_{X_k,T}$;
- All cycles in $G$ only contains label-synchronous elements.

**RP III:** Let $n = |\Delta^G|$. If the $lcs_T(C, D)$ exists, then

- $X^{n+1}$ is the $lcs_T(C, D)$, and
- $rd(lcs_T(C, D)) \leq 2^{2x|N_{C,T}|+1}$.

Future Works

- Practical implementation for the results above;
- Computing the $lcs$ w.r.t. general $\mathcal{FL}_0$-TBox;
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  - There is a $k \in \mathbb{N}$ s.t. $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T} = \mathcal{I}_{X^k,T}$;
  - There is a $k \in \mathbb{N}$ s.t. $(\mathcal{G},[\varepsilon]) \simeq \mathcal{J}_{X^k,T}$;
  - All cycles in $\mathcal{G}$ only contains label-synchronous elements.
- **RP III:** Let $n = |\Delta^G|$. If the $lcs_T(C,D)$ exists, then
  - $X^{n+1}$ is the $lcs_T(C,D)$, and
  - $rd(lcs_T(C,D)) \leq 2^{2^{|N_{C,T}|+1}}$.

Future Works

- Practical implementation for the results above;
- Computing the lcs w.r.t. general $\mathcal{FL}_E$-TBox;
- Computing the most specific concept of an individual w.r.t. general $\mathcal{FL}_0$-TBox.
Thank You