Unification in Description Logics

Part II: Unification in the DL $\mathcal{EL}$

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The lightweight DL $\mathcal{EL}$

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- Underlies the OWL 2 EL profile and can be used to define large biomedical ontologies, such as SNOMED CT.

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Characterization of subsumption

Corollary 5 [BM10b]

Let C and D be

$$C = A_1 \sqcap \ldots \sqcap A_k \sqcap \exists r_1.C_1 \sqcap \ldots \sqcap \exists r_m.C_m, \text{ and}$$
$$D = B_1 \sqcap \ldots \sqcap B_\ell \sqcap \exists s_1.D_1 \sqcap \ldots \sqcap \exists s_n.D_n, \text{ where}$$

$A_1, \ldots, A_k, B_1, \ldots, B_\ell \in \mathsf{N_C}$. Then $C \sqsubseteq D$ **iff** $\{B_1, \ldots, B_\ell\} \subseteq \{A_1, \ldots, A_k\}$ and for every j, $1 \leq j \leq n$, there exists an i, $1 \leq i \leq m$, such that $r_i = s_j$ and $C_i \sqsubseteq D_j$.

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- However, the green solution implies that $\mathcal{M}$ contains σ such that:

$$\sigma(X) \not\equiv \top \qquad \text{and} \qquad \sigma(X) \not\equiv \exists r.\top.$$

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- $\mathcal{M}$ contains σ of the form:

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there is $\tau \in \mathcal{M}$ s.t. $\tau \preceq \widehat{\sigma}$ $\quad \Rightarrow \quad$ $\tau \preceq \widehat{\sigma} \preceq \sigma$, $\tau \neq \sigma$

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Idea: reduce the propositional satisfiability problem to $\mathcal{EL}$-unification.

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SAT Problem

Instance:	A propositional formula φ in CNF: $\varphi = c_1 \wedge \ldots \wedge c_m$, where each c_i is a disjunction of literals.
Question:	Is there an assignment $t : \mathrm{Vars}(\varphi) \rightarrow \{\mathrm{t}, \mathrm{f}\}$ satisfying φ?

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Given φ, we build an $\mathcal{EL}$-unification problem Γ_φ such that:

φ is satisfiable **if, and only if**, Γ_φ has a unifier.

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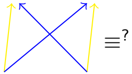

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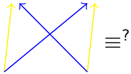

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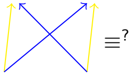

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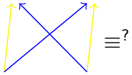

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Theorem 6 [BK00]

$\mathcal{EL}$-unification is NP-hard. Even for the special case of matching!

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Γ has a unifier **iff** Γ' is unifiable.

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Idea of "in NP" upper bound:

Every unifiable flat $\mathcal{EL}$-unification problem has a local unifier.

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Theorem 7 [BM10b, BBM12b]

Let Γ be a flat unification problem. If Γ has a unifier, then it also has a local unifier.

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Proof of Theorem 7 (Sketch)

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$$\theta(X_1) \sqsubseteq \exists r_2.\theta(X_2) \sqsubseteq \ldots \sqsubseteq \exists r_1 \ldots \exists r_n.\theta(X_1)$$

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  Not true! Hence, σ_{S^θ} is a local substitution.

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- It remains to show that σ_{S^θ} is a solution of Γ.

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We prove a more general claim.

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Lemma 8

Let $E_1, \ldots, E_m, D$ be atoms in Γ. Then,

$$\theta(E_1) \sqcap \ldots \sqcap \theta(E_m) \sqsubseteq \theta(D)$$

$$\Rightarrow$$

$$\sigma_{S^\theta}(E_1) \sqcap \ldots \sqcap \sigma_{S^\theta}(E_m) \sqsubseteq \sigma_{S^\theta}(D).$$

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Proof technique. Use induction on:

$$\max\{\mathrm{rd}(\sigma_{S^\theta}(G)) \mid G \in \{E_1, \ldots, E_m, D\}\},$$

together with the characterization of subsumption.

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Let $E_1, \ldots, E_m, D$ be atoms in Γ. Then,

$$\theta(E_1) \sqcap \ldots \sqcap \theta(E_m) \sqsubseteq \theta(D)$$

$$\Rightarrow$$

$$\sigma_{S^\theta}(E_1) \sqcap \ldots \sqcap \sigma_{S^\theta}(E_m) \sqsubseteq \sigma_{S^\theta}(D).$$

Proof technique. Use induction on:

$$\max\{\mathrm{rd}(\sigma_{S^\theta}(G)) \mid G \in \{E_1, \ldots, E_m, D\}\},$$

together with the characterization of subsumption.

rd = role-depth of a concept description.

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- Answer: seems unlikely!

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The minimal unifier containment problem:

Instance:	A flat $\mathcal{EL}$-unification problem Γ, a set $\mathcal{X} \subseteq \mathsf{N_v}$, a concept constant $A \in \mathsf{N_C}$ and a concept variable $X \in \mathcal{X}$.
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- NP=Σ_2^p $\Rightarrow$ the polynomial hierarchy collapses.

Unification Modulo $\mathcal{EL}$-TBoxes

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- What can we do? Consider a restricted class of TBoxes.

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Cycle-restricted TBoxes

A general TBox $\mathcal{T}$ is called cycle-restricted

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there is no word $w \in {\mathsf{N_R}}^+$ and $\mathcal{EL}$ concept C such that $C \sqsubseteq_\mathcal{T} \exists w.C$.

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- σ_{S^θ} is a local substitution.

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Theorem 9 [BBM12b]

Let $\mathcal{T}$ be a flat cycle-restricted TBox and Γ a flat unification problem. If Γ has a unifier w.r.t. $\mathcal{T}$, then it also has a local unifier w.r.t. $\mathcal{T}$.

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