Unification in Description Logics
Part IV: Related work in Modal Logics

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Basic modal systems
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Let $x_1, x_2, \ldots$ be propositional variables and $p_1, \ldots, p_m$ modal parameters.
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Basic modal propositional formulas

$$A, B ::= x \mid \top \mid \neg A \mid A \land B \mid \Box p A,$$

where $x$ is a propositional variable and $p$ a modal parameter.
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A set of formulas closed under substitutions such that it contains:
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A set of formulas closed under substitutions such that it contains:

- all classical tautologies (e.g. $\neg(x \land \neg x)$).
- the Aristotle axiom $\Box (x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$. 
Modal Logic (ML)

A formula $A$ is derivable in $L$ ($\vdash L A$) iff there is a sequence of formulas $B_1, ..., B_n = A$ such that:

- $B_i \in L$, or
- it can be obtained from previous elements in the sequence by applying the rules:
  - $x \rightarrow y$ (MP) or
  - $\Box x$ (necessitation).

The set of formulas which are derivable from the axiom system $L$.

Examples of modal logics:

- The minimum modal logic called K (with only one modal parameter).
- The logic K4: includes the axiom $\Box x \rightarrow \Box \Box x$.
- The logic S4: consists of K4 plus the axiom $\Box x \rightarrow x$.
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Modal Logic. Semantics

Kripke structures

- A Kripke frame is a pair \( F = (W, (R_{p1}, \ldots, R_{pn})) \) where:
  - \( W \) is a non-empty set of states (or possible worlds).
  - \( (R_{p1}, \ldots, R_{pn}) \) is a tuple of binary relations over \( W \) (accessibility relations).

- A Kripke model is a pair \( M = (F, V) \) where \( V \) is a valuation of the propositional variables:
  \[ V : \text{Vars} \rightarrow 2^{W} \]
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Validity

A is valid in a world $w$ of a model $M$ ($M, w \models A$) iff $M, w \models \top$.

$A \iff M, w \not\models A$.

$A \land B$ iff $M, w \models A$ and $M, w \models B$.

$\Box p A$ iff for all $w'$: $R_p(w, w') \Rightarrow M, w' \models A$.

$A$ is valid in a model $M$ ($M \models A$) iff it is valid in all its worlds.

$A$ is valid in a frame $F$ ($F \models A$) iff it is valid in all the models based on $F$.

$A$ is valid in a class of Kripke frames $K$ ($K \models A$) iff it is valid in all $F \in K$.

$L(K)$ is called the modal logic induced by the class of frames $K$. 
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Derivability vs Semantics (or $\vdash_L$ vs. validity)

In many cases $\vdash L$ corresponds to validity in a class of frames $K$.

- Minimum modal logic $K$: $\vdash K A$ iff $K \models A$ ($K$ is the class of all frames).
- Modal logic $K4$ ($\Box x \rightarrow \Box \Box x$): $\vdash K4 A$ iff $T \models A$ ($T$ is the class of all transitive frames).

There are modal logics that cannot be obtained from a class of Kripke frames [VB84].
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Correspondence with DLs [Sch91]

The DL $\mathcal{ALC}$ is a notational variant of $K_m$ (K plus $m$ modal parameters).

- Bijective translation between $\mathcal{ALC}$ concepts $C$ and $K_m$ formulas $A$.
  
  $A \rightarrow x \rightarrow$ modal parameter $p_i \forall r_i \rightarrow \Box p_i$

- Bijective translation between interpretations and Kripke models $I \rightarrow M I$ s.t:
  
  $A I = V M I (x A)$ and $(r_i) I = R p_i$.

- Inference problems $\mathcal{ALC}$ is valid in $K_m$ iff $C \equiv \top$ $C \equiv D$ iff $A \mathcal{ALC} \leftrightarrow A D$ is valid in $K_m$. 


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- Inference problems

  \[
  A_C \text{ is valid in } K_m \iff C \equiv \top
  \]

  \[
  C \equiv D \iff A_C \leftrightarrow A_D \text{ is valid in } K_m.
  \]
Let $L$ be a modal logic. The unification problem in $L$ is defined as follows.

**Instance:** A formula $A$ in $L$.

**Question:** Is there a substitution $\sigma$ such that $\vdash_L \sigma(A)$?
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- Unification type of $A$: defined w.r.t. $(U_L(A), \leq^{\text{Vars}(A)}_L)$. 
Unification - MLs vs DLs

Slightly different definitions:

ML: find $\sigma$ such that $\vdash L\sigma (A)$.

DLs: find $\sigma$ such that $\sigma (C) \equiv \sigma (D)$.

They "coincide" (if $\iff$ is expressible in the logic):

From ALC to $K_m$:
$\sigma (C) \equiv \sigma (D)$ iff $\vdash K_m \sigma (A\wedge C) \iff \sigma (A\wedge D)$ iff $\vdash K_m \sigma (A\wedge C \iff A\wedge D)$.

From $K_m$ to ALC:
$\vdash K_m \sigma (A)$ iff $\sigma (A) \equiv \top$.

Yet another subtle/significant difference

• For DLs, concept constants are allowed in the unification problem.
• For MLs, all variables are eligible to be substituted.
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• In DLs, \( \{C_1 \equiv ? D_1, \ldots, C_n \equiv ? D_n\} \) can be transformed into:
  \[ \{\forall r_1. C_1 \sqcap \ldots \sqcap \forall r_n. C_n \equiv ? \forall r_1. D_1 \sqcap \ldots \sqcap \forall r_n. D_n\} \].
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- In uni-modal logics, like K, the previous trick is not possible. However,
  \( \sigma \) solves \( \{A_1, \ldots, A_n\} \) iff it solves \( \{A_1 \land \ldots \land A_n\} \).
Motivation for unification in MLs

Unification in MLs is a special case of the recognizability of admissible rules problem.
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Recognizability of admissible rules

Instance: A modal logic $L$ and a rule $\frac{A}{B}$.

Question: Does $\vdash_L \sigma(A)$ implies $\vdash_L \sigma(B)$ for every substitution $\sigma$?

A positive answer means that $\frac{A}{B}$ can be added to $L$ without changing the logic.
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\]

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iff

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Some results

Positive results

• For K4, S₄ and other modal systems:
  • Unification is finitary and finite complete sets of unifiers can be computed.
  • Recognizability of admissible rules is decidable.

Negative results [WZ08]

• Undecidable for any modal logic L with universal modality between Kₜ₊ and K₄ₜ₊.
• Implies undecidability of unification in expressive and relevant DLs, like SHIQ.

Main open problem

• Unification and admissibility in K. K has unification type zero [Jer15]!
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Detect redundancies in ontologies

Sub-Boolean DLs

NP-c, ExpTime-c

arbitrary TBoxes

open problem

Restricted cases

same complexity

Modal Logic

Admissibility problem

Open problem for K - ALC

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