TECHNISCHE UNIVERSITÄT DRESDEN

**Faculty of Computer Science** ● Institute of Theoretical Computer Science ● Chair of Automata Theory

# REPAIRING $\mathcal{EL}$ TBOXES BY MEANS OF COUNTERMODELS OBTAINED BY MODEL TRANSFORMATION

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## Basics

- Description logic: $\mathcal{EL}$
- Concepts: $C ::= \top \mid A \mid C \sqcap C \mid \exists r.C$ where $A$ ranges over $\Sigma_{\mathsf{C}}$ and $r$ ranges over $\Sigma_{\mathsf{R}}$
- Concept inclusions: $C_1 \sqsubseteq C_2$
- TBox: finite set of concept inclusions
- Model-theoretic semantics

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- TBox: finite set of concept inclusions
- Model-theoretic semantics

Some notions:

- $\mathsf{Sub}(D)$ is the set of all subconcepts of $D$.
  Example: $\mathsf{Sub}(A_1 \sqcap \exists r.(B_1 \sqcap B_2)) = \{A_1 \sqcap \exists r.(B_1 \sqcap B_2),\ A_1,\ \exists r.(B_1 \sqcap B_2),\ B_1 \sqcap B_2,\ B_1,\ B_2\}$
- $\mathsf{Conj}(D)$ is the set of all top-level conjuncts in $D$.
  Example: $\mathsf{Conj}(A_1 \sqcap \exists r.(B_1 \sqcap B_2)) = \{A_1,\ \exists r.(B_1 \sqcap B_2)\}$
- $\mathsf{Atoms}(D)$ is the set of all atoms occurring in $D$.
  Example: $\mathsf{Atoms}(A_1 \sqcap \exists r.(B_1 \sqcap B_2)) = \{A_1,\ \exists r.(B_1 \sqcap B_2),\ B_1,\ B_2\}$

# Overview of our Approach

- Consider a TBox $\mathcal{T}$ that entails a concept inclusion $C \sqsubseteq D$ (unwanted consequence).
- **Goal:** compute a *repair*, i.e., a TBox $\mathcal{T}'$ where $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \not\models C \sqsubseteq D$.

## Overview of our Approach

- Consider a TBox $\mathcal{T}$ that entails a concept inclusion $C \sqsubseteq D$ (unwanted consequence).
- **Goal:** compute a *repair*, i.e., a TBox $\mathcal{T}'$ where $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \not\models C \sqsubseteq D$.
- We calculate a repair as follows.
  1. Construct the canonical model $\mathcal{I}_{C,\mathcal{T}}$.
  2. Transform $\mathcal{I}_{C,\mathcal{T}}$ to a countermodel $\mathcal{J}$ to $C \sqsubseteq D$.
  3. Axiomatize the logical intersection of $\mathcal{T}$ and $\mathcal{J}$.

## Countermodels

- **Definition:** A *countermodel* to $C \sqsubseteq D$ is an interpretation that does not satisfy $C \sqsubseteq D$.
- $\mathcal{I}$ is a countermodel to $C \sqsubseteq D$ if and only if there is $d \in \Delta^{\mathcal{I}}$ such that $d \in C^{\mathcal{I}}$ and $d \notin D^{\mathcal{I}}$.
- Thus: a countermodel to $C \sqsubseteq D$ exists if and only if $C \sqsubseteq D$ is no tautology.
- **Goal:** Construct a countermodel to the unwanted consequence $C \sqsubseteq D$ of $\mathcal{T}$ that does not invalidate too many other consequences of $\mathcal{T}$.

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- **Goal:** Construct a countermodel to the unwanted consequence $C \sqsubseteq D$ of $\mathcal{T}$ that does not invalidate too many other consequences of $\mathcal{T}$.
- As starting point, we use the canonical model $\mathcal{I}_{C,\mathcal{T}}$:

$$\Delta^{\mathcal{I}_{C,\mathcal{T}}} \coloneqq \{d_C\} \cup \{\, d_{C'} \mid \exists r.\, C' \in \mathsf{Sub}(C) \cup \mathsf{Sub}(\mathcal{T}) \,\}$$
$$A^{\mathcal{I}_{C,\mathcal{T}}} \coloneqq \{\, d_D \mid \mathcal{T} \models D \sqsubseteq A \,\}$$
$$r^{\mathcal{I}_{C,\mathcal{T}}} \coloneqq \{\, (d_D, d_{D'}) \mid \mathcal{T} \models D \sqsubseteq \exists r.\, D' \text{ and } \exists r.\, D' \in \mathsf{Sub}(\mathcal{T}) \cup \mathsf{Conj}(D) \,\}$$

- Recall that $\mathcal{I}_{C,\mathcal{T}} \models \mathcal{T}$ and $d_C \in C^{\mathcal{I}_{C,\mathcal{T}}}$. Thus $d_C \in D^{\mathcal{I}_{C,\mathcal{T}}}$.
- **Idea:** We will modify $\mathcal{I}_{C,\mathcal{T}}$ such that $d_C$ stays an instance of $C$ but is no instance of $D$ anymore.

# Countermodels

## Underlying Idea

- We adopt our technique for repairing ABoxes that removes unwanted consequences in form of concept assertions $D(a)$.

  Franz Baader, Francesco Kriegel, Adrian Nuradiansyah, and Rafael Peñaloza: *Computing Compliant Anonymisations of Quantified ABoxes w.r.t. $\mathcal{EL}$ Policies.* In Proceedings of the 19th International Semantic Web Conference (ISWC 2020).

- The construction of these ABox repairs is based on the negation of a recursive characterization of the instance problem:

  $a$ is no instance of $D$ w.r.t. $\mathcal{A}$ if and only if there is an atom $A$ or $\exists r.E$ in $\mathsf{Conj}(D)$ such that $A(a) \notin \mathcal{A}$ or each $r$-successor of $a$ is no instance of $E$ w.r.t. $\mathcal{A}$, respectively.

- Prior to removing assertions, we also create copies of objects in order to minimize the amount of information loss.

## Countermodels

Example

- TBox $\mathcal{T} \coloneqq \{A \sqsubseteq A_1,\ B \sqsubseteq B_1 \sqcap B_2\}$
- Unwanted consequence $C \sqsubseteq D \coloneqq A \sqcap \exists r.B \sqsubseteq A_1 \sqcap \exists r.(B_1 \sqcap B_2)$

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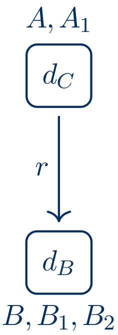
Canonical model $\mathcal{I}_{C,\mathcal{T}}$



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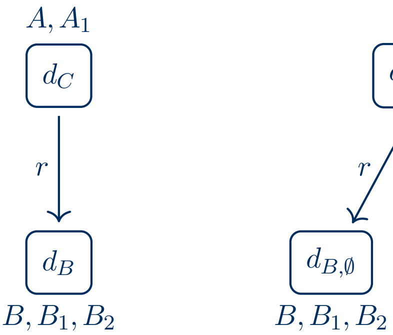

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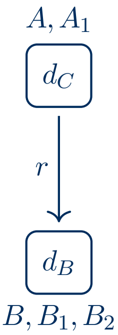
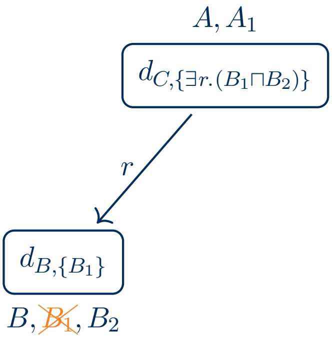



Countermodel $\mathcal{J}$



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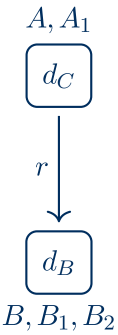


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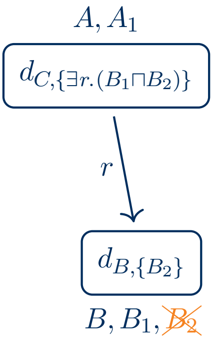

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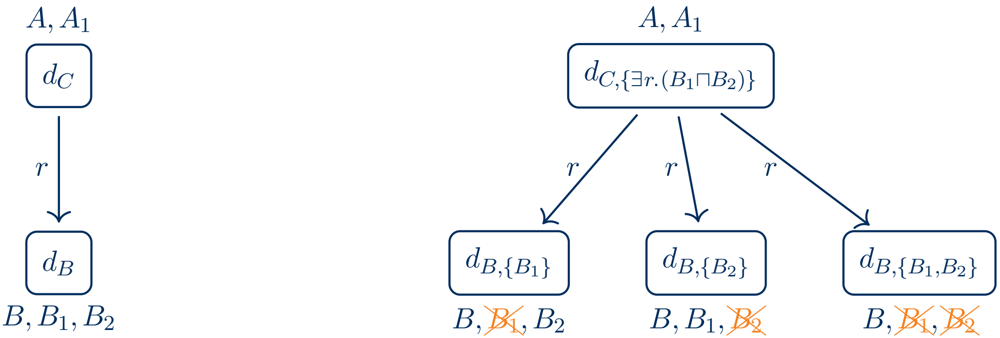

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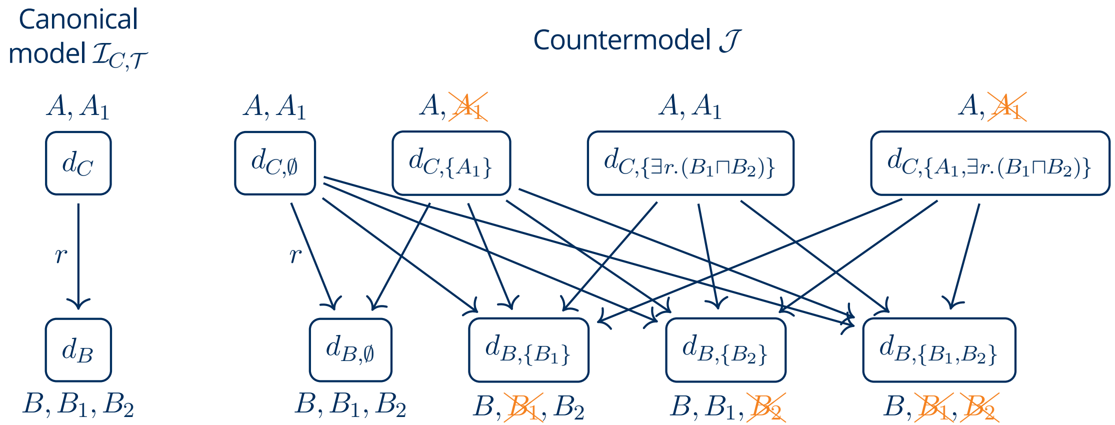

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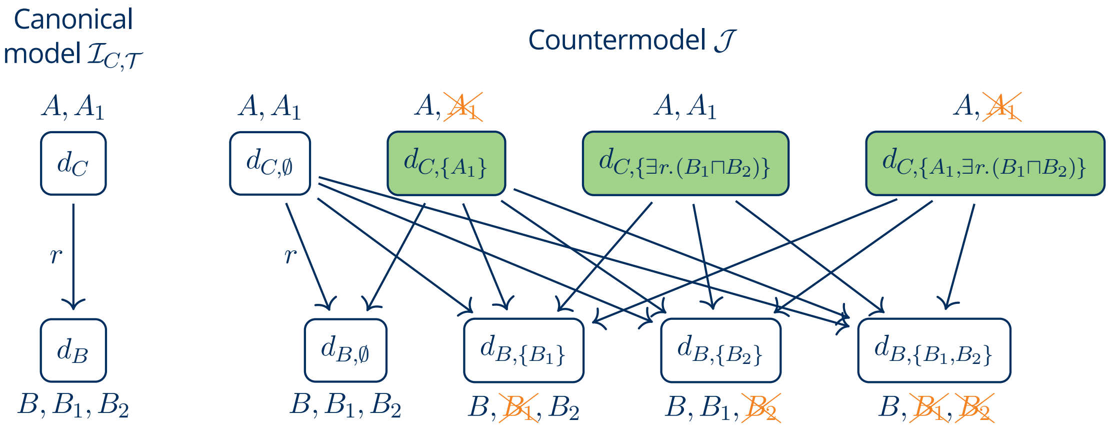

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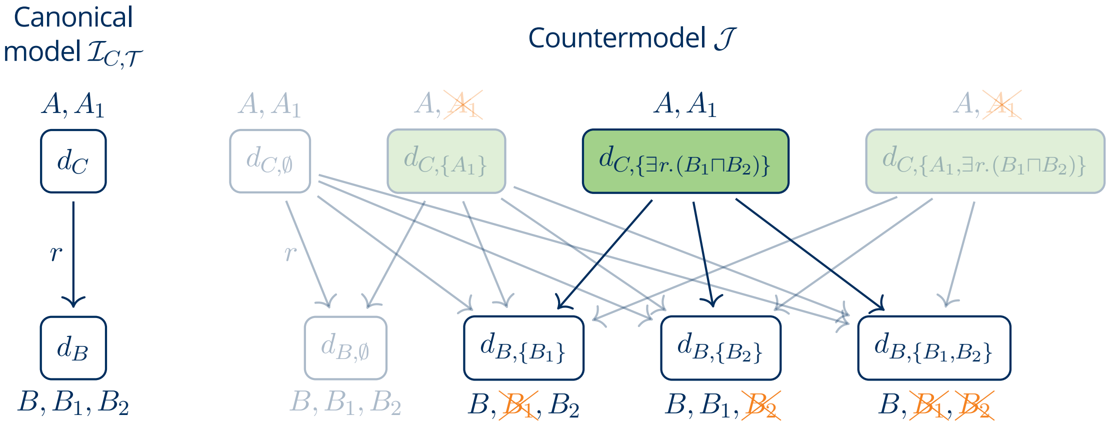

# Countermodels

## A Transduction based on ABox Repairs

- We use transductions as formalism for describing the transformation of $\mathcal{I}_{C,\mathcal{T}}$ to a countermodel.
- **Definition:** A *transduction* $\tau$ is a binary relation on interpretations.

# Countermodels

## A Transduction based on ABox Repairs

- We use transductions as formalism for describing the transformation of $\mathcal{I}_{C,\mathcal{T}}$ to a countermodel.
- **Definition:** A *transduction* $\tau$ is a binary relation on interpretations.
- Based on our ABox repair technique, the functional transduction $\tau_{\text{repair},D}$ maps each interpretation $\mathcal{I}$ to the following interpretation:

$$\Delta^{\tau_{\text{repair},D}(\mathcal{I})} := \left\{ d_{\mathcal{K}} \;\middle|\; \begin{array}{l} d \in \Delta^{\mathcal{I}},\ \mathcal{K} \subseteq \text{Atoms}(D),\ d \in F^{\mathcal{I}} \text{ for each } F \in \mathcal{K}, \\ \text{and } \mathcal{K} \text{ does not contain } \sqsubseteq_\emptyset\text{-comparable atoms} \end{array} \right\}$$

$$A^{\tau_{\text{repair},D}(\mathcal{I})} := \{\, d_{\mathcal{K}} \mid d \in A^{\mathcal{I}} \text{ and } A \notin \mathcal{K} \,\}$$

$$r^{\tau_{\text{repair},D}(\mathcal{I})} := \left\{ (d_{\mathcal{K}}, e_{\mathcal{L}}) \;\middle|\; \begin{array}{l} (d,e) \in r^{\mathcal{I}} \text{ and for each } \exists r.Q \in \mathcal{K} \text{ with } e \in Q^{\mathcal{I}}, \\ \text{there is } F \in \mathcal{L} \text{ such that } Q \sqsubseteq_\emptyset F \end{array} \right\}$$

- **Proposition:** $\tau_{\text{repair},D}(\mathcal{I}_{C,\mathcal{T}})$ is a countermodel to $C \sqsubseteq D$ and is computable in exponential time.

# Countermodels

## A Transduction that Extracts a Small Countermodel

- We have seen in the example that $\tau_{\mathsf{repair},D}(\mathcal{I}_{C,\mathcal{T}})$ contains several countermodels to $C \sqsubseteq D$ as subinterpretations.
- In order to extract these, we apply a second transduction $\tau_{\mathsf{reach}}$ to $\tau_{\mathsf{repair},D}(\mathcal{I}_{C,\mathcal{T}})$.
- This transduction $\tau_{\mathsf{reach}}$ is described by a monadic second-order definition scheme:

| | | |
|---|---|---|
| precondition formula | $\chi(\mathcal{W}) \coloneqq C^{\#}(v) \land \neg D^{\#}(v)$ | |
| domain formula | $\delta(\mathcal{W}, x) \coloneqq \mathit{reach}(v, x)$ | |
| concept formulae | $\theta_A(\mathcal{W}, x) \coloneqq A(x)$ | for each $A \in \Sigma_{\mathsf{C}}$ |
| role formulae | $\eta_r(\mathcal{W}, x, y) \coloneqq r(x, y)$ | for each $r \in \Sigma_{\mathsf{R}}$ |

  where $v$ is a first-order parameter in $\mathcal{W}$
  and $\mathit{reach}(x, y) \coloneqq \forall X\colon x \in X \land (\forall x, y\colon x \in X \land \bigvee_{r \in \Sigma_{\mathsf{R}}} r(x, y) \to y \in X) \to y \in X$
- **Proposition:** $\tau_{\mathsf{reach}}(\tau_{\mathsf{repair},D}(\mathcal{I}_{C,\mathcal{T}}))$ is a set of countermodels to $C \sqsubseteq D$ and is computable in exponential time.

# Logical Intersection

- Let $\mathcal{T}$ be a TBox and let $\mathcal{J}$ be a countermodel to the unwanted consequence $C \sqsubseteq D$.
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- **Definition:** The *logical intersection* $\mathcal{T} \vartriangle \mathcal{J}$ of $\mathcal{T}$ and $\mathcal{J}$ consists of all concept inclusions that both are entailed by $\mathcal{T}$ and are satisfied by $\mathcal{J}$.
- By definition, each concept inclusion in $\mathcal{T} \vartriangle \mathcal{J}$ is entailed by $\mathcal{T}$.
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- One might now think that $\mathcal{T} \vartriangle \mathcal{J}$ is a repair, but ...
- $\mathcal{T} \vartriangle \mathcal{J}$ is infinite in general, and so $\mathcal{T} \vartriangle \mathcal{J}$ cannot directly be used as a repair.
- **Idea:** Compute a finite axiomatization of $\mathcal{T} \vartriangle \mathcal{J}$.

# Logical Intersection

## Axiomatization

- To axiomatize $\mathcal{T} \mathbin{\triangle} \mathcal{J}$, we utilize the approach described in:
  Francesco Kriegel: *Constructing and Extending Description Logic Ontologies using Methods of Formal Concept Analysis.* Doctoral Thesis, Technische Universität Dresden, 2019.

In a nutshell:

- A *closure operator* (clop) $\varphi$ maps each concept $E$ to a concept $E^\varphi$ such that
  1. $E^\varphi \sqsubseteq_\emptyset E$,
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- A concept inclusion $E \sqsubseteq F$ is *valid* for $\varphi$ if $E^\varphi \sqsubseteq_\emptyset F$, written $\varphi \models E \sqsubseteq F$.

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- A concept inclusion $E \sqsubseteq F$ is *valid* for $\varphi$ if $E^\varphi \sqsubseteq_\emptyset F$, written $\varphi \models E \sqsubseteq F$.
- The TBox $\mathcal{T}$ induces a clop $\varphi_\mathcal{T}$ such that $\mathcal{T} \models E \sqsubseteq F$ if and only if $\varphi_\mathcal{T} \models E \sqsubseteq F$.
- The countermodel $\mathcal{J}$ induces a clop $\varphi_\mathcal{J}$ such that $\mathcal{J} \models E \sqsubseteq F$ if and only if $\varphi_\mathcal{J} \models E \sqsubseteq F$.
- The infimum $\varphi_\mathcal{T} \vartriangle \varphi_\mathcal{J}$ where $E^{\varphi_\mathcal{T} \vartriangle \varphi_\mathcal{J}} = \operatorname{lcs}(E^{\varphi_\mathcal{T}}, E^{\varphi_\mathcal{J}})$ has the following important property: $\varphi_\mathcal{T} \vartriangle \varphi_\mathcal{J} \models E \sqsubseteq F$ if and only if $\varphi_\mathcal{T} \models E \sqsubseteq F$ and $\varphi_\mathcal{J} \models E \sqsubseteq F$.
- The canonical base of $\varphi_\mathcal{T} \vartriangle \varphi_\mathcal{J}$ is an axiomatization of $\mathcal{T} \vartriangle \mathcal{J}$.

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- TBox $\mathcal{T} :=$ $\{A \sqsubseteq A_1,\ B \sqsubseteq B_1 \sqcap B_2\}$
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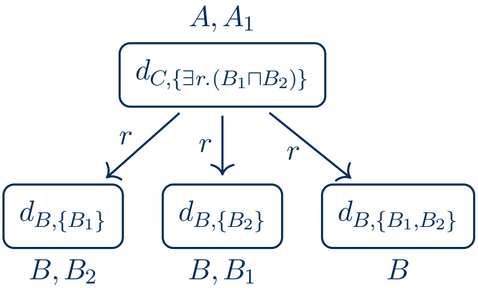

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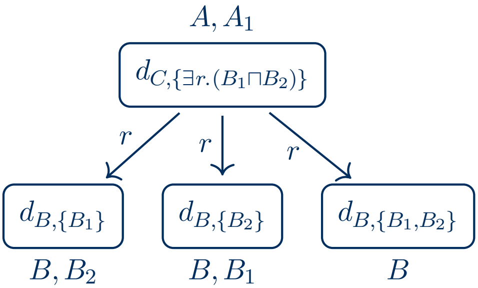


Axiomatization of $\mathcal{T} \vartriangle \mathcal{J}$:

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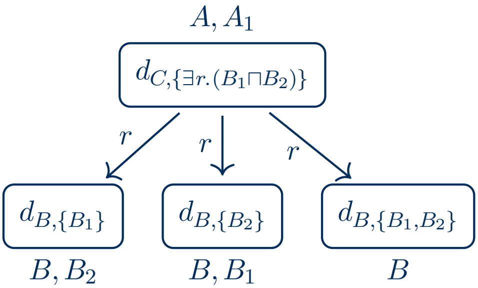


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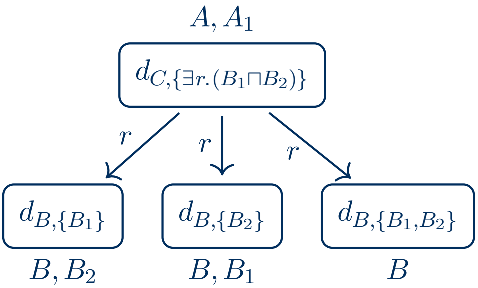


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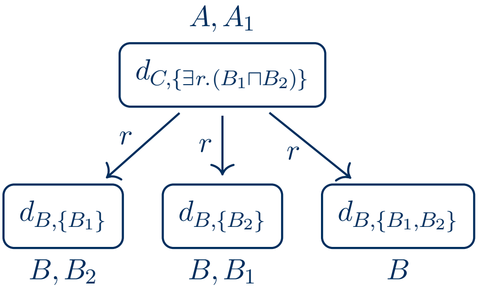


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4. $\exists s.B \sqsubseteq \exists s.(B \sqcap B_1 \sqcap B_2)$ where $s \in \Sigma_R \setminus \{r\}$

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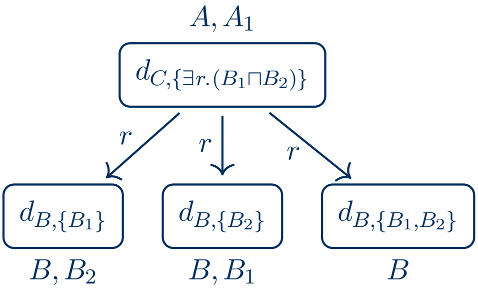


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5. $\exists r.(B \sqcap B_1) \sqcap \exists r.(B \sqcap B_2) \sqcap E \sqsubseteq \exists r.(B \sqcap B_1 \sqcap B_2)$
   - where $E \in \Sigma_{\mathsf{C}} \setminus \{A, A_1, B\}$
   - or $E = \exists r.F$ for $F \in \Sigma_{\mathsf{C}} \setminus \{A, B, B_1, B_2\}$
   - or $E = \exists r.(B_1 \sqcap B_2)$
   - or $E = \exists s.\top$ for $s \in \Sigma_{\mathsf{R}} \setminus \{r\}$
6. $\ldots$

That’s it for now!

Do you have questions or comments?