Lemmas and theorems

Foundations of Logic Programming

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1 Subsitutions

1. We say that a substitution γ is a *renaming* if γ is $1-1$ and *onto* mapping from its domain to itself (i.e. a *permutation* of the domain).

   Prove that for every renaming θ there exists only one substitution θ^{-1} such that $\theta\theta^{-1} = \theta^{-1}\theta = \epsilon$. Prove that θ^{-1} is a renaming of θ.

2. s is called an *instance* of t if $s = t\sigma$ for a substitution σ. Prove that s is a variant of t iff s is an instance of t and t is an instance of s.

3. **Renaming Lemma**: $\theta \leq \eta$ and $\eta \leq \theta$ iff there is γ a renaming of θ such that, $\eta = \theta\gamma$.

2 Unification

1. **Binding Lemma**: For a variable x and a term t, $x\theta = t\theta$ iff $\theta = \{x \mapsto t\}\theta$.

2. **Solved Form Lemma**: If $E := \{x_1 = t_1, \ldots, x_n = t_n\}$ is solved, then the substitution $\theta := \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ is an mgu of E. θ is idempotent.

3. Prove that for each $n \geq 1$, $\succ_n$ (lexicographic ordering defined on n-tuples of natural numbers) is a well-founded order.

4. The Martelli-Montanari algorithm always terminates.

5. Each step of the Martelli-Montanari algorithm replaces the set of equations by an equivalent one.

6. If the Martelli-Montanari algorithm terminates with success, then the final set of equations is solved.

7. If the Martelli-Montanari algorithm terminates with failure, then the set of equations at the moment of failure does not have a unifier.

8. **Unification Theorem** (correctness of the Martelli-Montanari algorithm): The Martelli-Montanari algorithm always terminates. If the original set of equations E has a unifier, then the algorithm terminates with success and produces a solved set of equations determining an mgu of E. Otherwise it terminates with failure.

9. **Idempotence Theorem**: An mgu is strong iff it is idempotent.

10. A substitution θ is idempotent iff $Dom(\theta) \cap VRan(\theta) = \emptyset$, where $VRan(\theta)$ is the set of variables in the range of θ.

11. Suppose that θ and η are idempotent substitutions such that $Dom(\theta) \cap VRan(\eta) = \emptyset$. Prove that $\theta\eta$ is idempotent.

12. **Equivalence Lemma**: Let θ_1 be an mgu of a set of equations E. Then for every substitution θ_2, θ_2 is an mgu of E iff $\theta_2 = \theta_1\gamma$ for a renaming γ.

13. **Relevance Theorem**: Every idempotent mgu is relevant.

14. **Iteration Lemma**: Let E_1, E_2 be two sets of equations. Suppose that θ_1 is an mgu of E_1 and θ_2 an mgu of $E_2\theta_1$. Then $\theta_1\theta_2$ is an mgu of $E_1 \cup E_2$. Moreover if $E_1 \cup E_2$, then an mgu θ_1 of E_1 exists and for any mgu θ_1 of E_1, an mgu θ_2 of $E_2\theta_1$ exists.

15. **Switching Corollary** (corollary to Iteration Lemma): Let E_1, E_2 be two sets of equations. Suppose that θ_1 is an mgu of E_1 and θ_2 an mgu of $E_2\theta_1$. Then E_2 is unifiable and for every mgu θ_1' of E_2 there is an mgu θ_2' of $E_1\theta_1'$ such that $\theta_1\theta_2 = \theta_1'\theta_2'$ and $Var(\theta_2') \subseteq Var(E_1) \cup Var(\theta_1') \cup Var(\theta_1\theta_2)$.

3 SLD-derivations

1. **Disjointness Lemma**: Consider an SLD-derivation of $P \cup \{Q\}$ with the sequence $d_1, \ldots, d_n, \ldots$ of input clauses used and with the sequence of $R_0, \ldots, R_n, \ldots$ of resultants associated with it. Then for $i \geq 0$, $Var(R_i) \cap Var(R_{i+1}) = \emptyset$.

2. **Propagation Lemma**: Suppose that $R \Rightarrow_c^{\theta} R_1$ and $R' \Rightarrow_c^{\theta'} R_1'$ are two SLD-resultant steps such that

   - R is an instance of R'
   - in R and R' atoms in the same positions are selected.

   Then R_1 is an instance of R_1'.

3. **Propagation Corollary**: Suppose that $Q \Rightarrow_c^{\theta} Q_1$ and $Q' \Rightarrow_c^{\theta'} Q_1'$ are two SLD-derivation steps such that

   - Q is an instance of Q'
   - in Q and Q' atoms in the same positions are selected.

Then Q_1 is an instance of Q'_1.

4. **Instance Theorem**: Consider an SLD-derivation ξ and its lift ξ'. Then for $i \geq 0$, if the resultant R_i of level i of ξ exists, then so does the resultant R'_i of level i of ξ' and R_i is an instance of R'_i.

5. **Variant Theorem**: Consider two similar SLD-derivations. Then for $i \geq 0$ their resultants of level i are variants of each other.

6. **Variant Corollary**: Consider two similar SLD-derivations of Q with c.a.s.s θ and η. Then $Q\theta$ and $Q\eta$ are variants of each other.

7. **Selection Note**: Every SLD-derivation is via selection rule.

8. **Switching Lemma**: Consider a query Q_n with two different atoms A_1 and A_2. Suppose that
   $\xi := Q_0 \Rightarrow^{\theta_1}_{c_1} Q_1 \dots Q_n \Rightarrow^{\theta_{n+1}}_{c_{n+1}} Q_{n+1} \Rightarrow^{\theta_{n+2}}_{c_{n+2}} Q_{n+2} \dots$
   is an SLD-derivation where:

   - A_1 is the selected atom of Q_n,
   - $A_2\theta_{n+1}$ is the selected atom of Q_{n+1}.

   Then for some $Q'_{n+1}, \theta'_{n+1}, \theta'_{n+2}$:

   - $\theta'_{n+1}\theta'_{n+2} = \theta_{n+1}\theta_{n+2}$
   - there exists an SLD-derivation:
     $\xi' := Q_0 \Rightarrow^{\theta_1}_{c_1} Q_1 \dots Q_n \Rightarrow^{\theta'_{n+1}}_{c_{n+2}} Q_{n+1} \Rightarrow^{\theta_{n'+2}}_{c_{n+1}} Q_{n+2} \dots$
     where
     - ξ and ξ' coincide up to the resolvent Q_n,
     - A_2 is the selected atom in Q_n
     - $A_1\theta'_{n+1}$ is the selected atom in Q'_{n+1},
     - ξ and ξ' conicide after the resolvent Q_{n+2}.

9. **Independence Theorem**: For every successful SLD-derivation ξ of $P \cup \{Q\}$ and a selection rule R, there exists a successful SLD-derivation ξ' of $P \cup \{Q\}$ via R such that:

   - the c.a.s.s of ξ and ξ' are the same,
   - ξ and ξ' are of the same length.

10. Every SLD-tree is via a variant independent selection rule.

11. **Branch Theorem**: Consider an SLD-tree $\mathcal{T}$ for $P \cup \{Q\}$ via a variant independent selection rule R. Then every SLD-derivation of $P \cup \{Q\}$ via R is similar to a branch of $\mathcal{T}$.

12. **Independence Corollary**: If an SLD-tree for $P \cup \{Q\}$ is successful, then all SLD-trees for $P \cup \{Q\}$ are successful.

4 Soundness and Completeness of SLD-resolution

1. **Resultant Lemma**

   (a) Let $Q \Rightarrow_c^\theta Q_1$ be an SLD-derivation step and r the resultant associated with it. Then $c \models r$.

   (b) Consider an SLD-derivation of $P \cup \{Q\}$ with the sequence $R_0, \ldots, R_n, \ldots$ of resultants associated with it. Then for all $i \geq 0$ $P \models R_i$.

2. **Soundness of SLD-resolution (theorem).** Suppose that there exists a successful SLD-derivation of $P \cup \{Q\}$ with c.a.s. θ. Then $P \models Q\theta$.

3. **Soundness of SLD-resolution (corollary).** Suppose that there exists a successful SLD-derivation of $P \cup \{Q\}$. Then $P \models \exists Q$.

4. **Term Interpretation Lemma.** Let I be a term interpretation. Then

   (a) for an atom A and a valuation (state) σ, $I \models_\sigma A$ iff $A(\sigma|Var(A)) \in I$,

   (b) for an atom A, $I \models A$ iff $inst(A) \subseteq I$,

   (c) for a clause c, $I \models c$ iff for all $A \leftarrow B_1, \ldots, B_n$ in $inst(c)$, $\{B_1, \ldots, B_n\} \subseteq I$ implies $A \in I$

5. **Substiution Closure Note.** For a term interpretation I closed under substitution, $I \models \exists Q$ implies that for some substitution θ, $I \models Q\theta$.

6. $C(P)$ **Lemma.** The term interpretation
   $C(P) := \{A \mid A \text{ has an implication tree w.r.t.} P\}$
   is a model of P.

7. Prove that $C(P)$ is closed under substitution.

8. **Implication Tree Lemma.** Suppose that $Q\theta$ is n-deep for some $n \geq 0$. Then for every selection rule R there exists a successful SLD-derivation of $P \cup \{Q\}$ via R with the c.a.s. η such that $Q\eta$ is more general than $Q\theta$.

9. **Strong Completeness of SLD-resolution Theorem.** Suppose that $P \models Q\theta$. Then for every selection rule R there exists a successful SLD-derivation of $P \cup \{Q\}$ via R with the c.a.s. η such that $Q\eta$ is more general than $Q\theta$.

10. **Completeness Corollary**. Suppose that $P \models \exists Q$. Then there exists a successful SLD-derivation of $P \cup \{Q\}$.