About Subsumption in Fuzzy $\mathcal{EL}^{\star}$

Stefan Borgwardt[1] and Rafael Peñaloza[1,2]

[1] Theoretical Computer Science, TU Dresden, Germany
[2] Center for Advancing Electronics Dresden
{stefborg,penaloza}@tcs.inf.tu-dresden.de

1 Introduction

Classical Description Logics (DLs) [2] cannot properly deal with the endemic imprecision of biomedical knowledge. For example, the current version of the SNOMED CT ontology defines a "Perinatal Cyanotic Attack" as a cardiovascular disorder occurring in the perinatal period and manifested through cyanosis. This definition depends on two vague notions, namely the *perinatal period*—the period of time around birth—and *cyanosis*—a bluish discoloration of the skin. While it is possible to say that one year after birth is not perinatal, and a few hours from birth is, there is no precise threshold on the end of the perinatal period. However, it makes sense to say that every child is *less* in its perinatal period as time goes by. A similar consideration can be made for skin turning from red to blue in cases of cyanosis. The use of several *degrees of truth* has been proposed for dealing with these gradual changes, as well as other kinds of imprecisions.

Mathematical Fuzzy Logic [12] generalizes classical logic by allowing real numbers from the interval $[0, 1]$ to act as truth degrees. It allows to express, e.g. that a newborn child is in the perinatal period with degree 1, but a three-week-old belongs to this period only with degree 0.3. In Mathematical Fuzzy Logic, the interpretation of the logical constructors, such as conjunction, disjunction, and implication, is determined by the choice of a binary *triangular norm* (or t-norm). Fuzzy Description Logics combine DLs with Mathematical Fuzzy Logic as a means to formally represent and reason with vague conceptual knowledge [18,19]. So far, research on fuzzy DLs was mainly focused on fuzzy extensions of propositionally closed DLs. Unfortunately, in fuzzy DLs a negation constructor often leads to undecidability [7,11].

To the best of our knowledge, the only fuzzy extensions of $\mathcal{EL}$ studied so far are based on the Gödel t-norm [16,20]. In these logics, fuzzy subsumption between concepts can be decided in polynomial time. Beyond this tractable case, very little is known about the complexity of subsumption with general t-norms. If we restrict the set of membership degrees to be finite, subsumption can be decided in exponential time [3,8], but for the interval $[0, 1]$ nothing is known, even for expressive fuzzy DLs in which consistency is decidable [5].

* Partially supported by the DFG under grant BA 1122/17-1, in the research training group 1763 (QuantLA), and in the Cluster of Excellence 'cfAED'

Table 1. The three fundamental continuous t-norms.

Name	t-norm ($x \otimes y$)	residuum ($x \Rightarrow y$)
Gödel	$\min\{x, y\}$	$\begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Product	$x \cdot y$	$\begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Łukasiewicz	$\max\{x + y - 1, 0\}$	$\min\{1 - x + y, 1\}$

We consider fuzzy extensions of $\mathcal{EL}$ with general t-norm semantics, and study their complexity. As for the classical case, we are interested in deciding subsumption between concepts. We study the problem of 1-subsumption, which can be seen as deciding classical subsumption between fuzzy concepts. We show that this problem is co-NP-hard in general for a wide variety of t-norms. However, if we restrict to normalized TBoxes, then under some additional assumptions this problem can be solved in polynomial time. To show this, we provide a completion-based algorithm that classifies the TBox w.r.t. 1-subsumption.

2 Preliminaries

We introduce the fuzzy DL $\otimes$-$\mathcal{EL}$ and its reasoning tasks, along with some of the properties that will be used throughout the paper. The semantics of $\otimes$-$\mathcal{EL}$ depends on the choice of a t-norm $\otimes$. A *t-norm* is an associative, commutative, and monotone binary operator $\otimes\colon [0,1] \times [0,1] \to [0,1]$ that has unit 1 [15]. We consider only *continuous* t-norms throughout this paper. Given a t-norm $\otimes$ and $x \in [0,1]$, we define $x^n := \bigotimes_{i=1}^{n} x$. Every continuous t-norm defines a unique *residuum* $\Rightarrow\colon [0,1] \times [0,1] \to [0,1]$ where $x \Rightarrow y := \sup\{z \mid x \otimes z \leq y\}$. From this it follows that (i) $x \Rightarrow y = 1$ iff $x \leq y$, and (ii) $1 \Rightarrow y = y$ hold for all $x, y \in [0,1]$. Table 1 lists three important continuous t-norms and their residua. All other continuous t-norms can be built as the ordinal sums of copies of these t-norms, as follows.

Let $((a_i, b_i))_{i \in I}$ be a (possibly infinite) family of non-empty, disjoint open subintervals of $[0,1]$ and $(\otimes_i)_{i \in I}$ be a family of continuous t-norms over the same index set I. The *ordinal sum* of $(((a_i, b_i), \otimes_i))_{i \in I}$ is the t-norm $\otimes$, where

$$x \otimes y := a_i + (b_i - a_i)\left(\tfrac{x - a_i}{b_i - a_i} \otimes_i \tfrac{y - a_i}{b_i - a_i}\right)$$

if $x, y \in [a_i, b_i]$ for some $i \in I$, and $x \otimes y := \min\{x, y\}$ otherwise. This yields a continuous t-norm, whose residuum $x \Rightarrow y$ is given by

$$\begin{cases} 1 & \text{if } x \leq y, \\ a_i + (b_i - a_i)\left(\tfrac{x - a_i}{b_i - a_i} \Rightarrow_i \tfrac{y - a_i}{b_i - a_i}\right) & \text{if } a_i \leq y < x \leq b_i, \\ y & \text{otherwise,} \end{cases}$$

where $\Rightarrow_i$ is the residuum of $\otimes_i$, for each $i \in I$ [15]. Intuitively, this means that the t-norm $\otimes$ and its residuum "behave like" $\otimes_i$ and its residuum in each of the intervals $[a_i, b_i]$, and like the Gödel t-norm and residuum everywhere else.

Theorem 1 ([17]). *Every continuous t-norm is isomorphic to the ordinal sum of copies of the Łukasiewicz and product t-norms.*

Let $\otimes$ be a continuous t-norm and $(((a_i, b_i), \otimes_i))_{i\in I}$ be its representation as ordinal sum given by Theorem 1.[3] We call $(((a_i, b_i), \otimes_i))_{i\in I}$ the *components* of $\otimes$. We say that $\otimes$ *contains* a t-norm $\otimes'$ if it has a component of the form $((a_i, b_i), \otimes')$. It *starts with Łukasiewicz* if it has a component of the form $((0, b), \otimes_{Ł})$, where $\otimes_{Ł}$ is the Łukasiewicz t-norm, and analogously for *ends with Łukasiewicz*. The only elements $x \in [0, 1]$ that are *idempotent* w.r.t. $\otimes$, i.e. that satisfy $x \otimes x = x$, are those that are not in (a_i, b_i) for any $i \in I$. Every continuous t-norm except the Gödel t-norm has infinitely many non-idempotent elements.

Every continuous t-norm $\otimes$ defines a fuzzy DL $\otimes$-$\mathcal{EL}$. If $\otimes$ is the Gödel or Łukasiewicz t-norm, we write G-$\mathcal{EL}$ or $\mathsf{Ł}$-$\mathcal{EL}$, respectively. The syntax of $\otimes$-$\mathcal{EL}$ is the same as in classical $\mathcal{EL}$. Concepts are built from two disjoint sets $\mathsf{N_C}$ and $\mathsf{N_R}$ of *concept* and *role names*, respectively, using the constructors top ($\top$), conjunction ($C_1 \sqcap C_2$), and existential restriction ($\exists r.C$). C^n denotes the n-ary conjunction of a $\otimes$-$\mathcal{EL}$-concept C with itself; $C^n := \bigsqcap_{i=1}^{n} C$. A $\otimes$-$\mathcal{EL}$-*TBox* is a finite set of *general concept inclusion axioms* (GCIs) of the form $\langle C \sqsubseteq D \geq q \rangle$, where C, D are $\otimes$-$\mathcal{EL}$-concepts and $q \in [0, 1]$. A $\otimes$-$\mathcal{EL}$-TBox is *crisp* all its GCIs are of the form $\langle C \sqsubseteq D \geq 1 \rangle$. We often drop the prefix $\otimes$-$\mathcal{EL}$ and speak simply of, e.g. concepts and TBoxes.

The semantics of this logic extends the classical DL semantics by interpreting concepts and roles as fuzzy sets and fuzzy binary relations, respectively, over an interpretation domain. Given a domain Δ, a *fuzzy set* is a function $F \colon \Delta \to [0, 1]$. Intuitively, an element $\delta \in \Delta$ belongs to the fuzzy set F *with degree* $F(\delta)$. An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty *domain*, and the interpretation function $\cdot^{\mathcal{I}}$ maps concept names A and role names r to functions $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0, 1]$ and $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$, respectively. The interpretation function is extended to $\otimes$-$\mathcal{EL}$-concepts by setting, for every $\delta \in \Delta$, $\top^{\mathcal{I}}(\delta) := 1$, $(C_1 \sqcap C_2)^{\mathcal{I}}(\delta) := C_1^{\mathcal{I}}(\delta) \otimes C_2^{\mathcal{I}}(\delta)$, and $(\exists r.C)^{\mathcal{I}}(\delta) := \sup_{\gamma \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(\delta, \gamma) \otimes C^{\mathcal{I}}(\gamma)$. An interpretation $\mathcal{I}$ *satisfies* the GCI $\langle C \sqsubseteq D \geq q \rangle$ iff $(C^{\mathcal{I}}(\delta) \Rightarrow D^{\mathcal{I}}(\delta)) \geq q$ for all $\delta \in \Delta^{\mathcal{I}}$. It is a *model* of the TBox $\mathcal{T}$ if it satisfies all the GCIs in $\mathcal{T}$. An interpretation $\mathcal{I}$ is called *crisp* if $A^{\mathcal{I}}(\delta) \in \{0, 1\}$ and $r^{\mathcal{I}}(\delta, \gamma) \in \{0, 1\}$ hold for every concept name A, role name r, and $\delta, \gamma \in \Delta^{\mathcal{I}}$.

Example 2. The concept of perinatal cyanotic attacks (PCA) can be described using the GCI

$$\langle \mathsf{PCA} \sqsubseteq \mathsf{CardiovascDisorder} \sqcap \exists \mathsf{occur.PerinatalPeriod} \sqcap \exists \mathsf{manif.Cyanosis} \geq 1 \rangle,$$

which is very close to the definition found in SNOMED CT. With the Łukasiewicz t-norm, an element that belongs to each of the concepts on the right-hand side

[3] For ease of presentation, we treat the isomorphism as equality.

with degree 0.7 will belong to PCA with degree at most $0.7+0.7+0.7-2=0.1$. While this makes sense from a diagnostic point of view—lesser symptomatic manifestations should yield a weaker diagnosis—SNOMED CT is meant to *describe* clinical terms, rather than diagnose them. It thus makes sense to divide the previous GCI into the three axioms

$$\langle \mathsf{PCA} \sqsubseteq \mathsf{CardiovascDisorder} \geq 1\rangle, \langle \mathsf{PCA} \sqsubseteq \exists \mathsf{occur}.\mathsf{PerinatalPeriod} \geq 1\rangle,$$
$$\langle \mathsf{PCA} \sqsubseteq \exists \mathsf{manif}.\mathsf{Cyanosis} \geq 1\rangle.$$

In fuzzy DLs, reasoning is sometimes restricted to *witnessed* interpretations [13]: interpretations $\mathcal{I}$ in which there is a $\gamma \in \Delta^{\mathcal{I}}$ with $(\exists r.C)^{\mathcal{I}}(\delta) = r^{\mathcal{I}}(\delta,\gamma) \otimes C^{\mathcal{I}}(\gamma)$. This restriction was introduced in [13] to correct the existing algorithm for fuzzy $\mathcal{ALC}$ in [19]. In this paper we do not need this additional assumption; all our results are valid w.r.t. general *and* witnessed semantics.

As in classical $\mathcal{EL}$, every $\otimes$-$\mathcal{EL}$-TBox has the trivial model $\mathcal{I} = (\{\delta\}, \cdot^{\mathcal{I}})$ where $A^{\mathcal{I}}(\delta) = 1$ for every concept name A and $r^{\mathcal{I}}(\delta,\delta) = 1$ for every role name r. Thus, TBox *consistency* is trivial in this logic. We are therefore interested in deciding subsumption between two concepts, and other related problems.

Definition 3. *Let $\mathcal{T}$ be a TBox, C, D be two concepts, and $p \in (0,1]$. C is* p-subsumed *by D w.r.t. $\mathcal{T}$ ($C \sqsubseteq^p_{\mathcal{T}} D$) if every model of $\mathcal{T}$ satisfies $\langle C \sqsubseteq D \geq p\rangle$. C is* positively subsumed *by D w.r.t. $\mathcal{T}$ ($C \sqsubseteq^{>0}_{\mathcal{T}} D$) if every model $\mathcal{I}$ of $\mathcal{T}$ and every $\delta \in \Delta^{\mathcal{I}}$ satisfies $C^{\mathcal{I}}(\delta) \Rightarrow D^{\mathcal{I}}(\delta) > 0$. The* best subsumption degree *of $C \sqsubseteq D$ w.r.t. $\mathcal{T}$ is $\mathsf{bsd}_{\mathcal{T}}(C \sqsubseteq D) := \sup\{p \in [0,1] \mid C \sqsubseteq^p_{\mathcal{T}} D\}$.*

Clearly, if $\mathsf{bsd}_{\mathcal{T}}(C \sqsubseteq D) > 0$, then $C \sqsubseteq^{>0}_{\mathcal{T}} D$. However, the converse does not necessarily hold, as evidenced by the following example.

Example 4. Consider the product t-norm and $A \in \mathsf{N_C}$. For every interpretation $\mathcal{I}$ and $\delta \in \Delta^{\mathcal{I}}$, if $A^{\mathcal{I}}(\delta) > 0$, then $A^{\mathcal{I}}(\delta) \Rightarrow (A^2)^{\mathcal{I}}(\delta) = A^{\mathcal{I}}(\delta) > 0$. Thus A is positively subsumed by A^2. However, for every $p > 0$ there is an interpretation $\mathcal{I} = (\{\delta\}, \cdot^{\mathcal{I}})$ with $A^{\mathcal{I}}(\delta) = p/2$. Then, $A^{\mathcal{I}}(\delta) \Rightarrow (A^2)^{\mathcal{I}}(\delta) = A^{\mathcal{I}}(\delta) = p/2 < p$. As this holds for every $p > 0$, it follows that $\mathsf{bsd}_{\emptyset}(A \sqsubseteq A^2) = 0$.

3 Hardness Results

In this section we show several hardness results for the decision problems that we have defined before. In particular, we describe families of t-norms for which deciding positive subsumption and 1-subsumption, as well as computing the best subsumption degree is not tractable (unless P = NP). We first show that 1-subsumption is co-NP-hard for the Łukasiewicz t-norm, by reducing the NP-hard vertex cover problem [14] to its complement.

Definition 5. *Let $V = \{v_1, \ldots, v_m\}$ be a finite set, and $\mathcal{E}$ a set of subsets of V of cardinality 2. A* vertex cover *is a set $S \subseteq V$ such that $S \cap E \neq \emptyset$ holds for all $E \in \mathcal{E}$. The* vertex cover problem *consists in deciding, given a natural number $k \leq m$, whether there is a vertex cover of cardinality $\leq k$.*

Every superset of a vertex cover is also a vertex cover, and thus one can equivalently ask for a vertex cover of size exactly k. We assume without loss of generality that the graph $(V, \mathcal{E})$ has no isolated nodes since such nodes are irrelevant for vertex covers. Given an instance $\mathcal{V} := (V, \mathcal{E}, k)$ of the vertex cover problem, we construct an Ł-$\mathcal{EL}$-TBox $\mathcal{T}_\mathcal{V}$ and two concept names A, B such that A is *not* 1-subsumed by B w.r.t. $\mathcal{T}_\mathcal{V}$ iff there is a vertex cover of size k. Let $V_i, 0 \leq i \leq m$, be concept names, where $m = |V|$, i.e. we have a concept name V_i for every $v_i \in V$, and an additional concept name V_0. For each $i, 1 \leq i \leq m$, we set

$$\mathcal{T}_i := \{\langle V_i^{m-k} \sqsubseteq V_i^{m-k+1} \geq 1\rangle,\ \langle \top \sqsubseteq V_i \geq \tfrac{m-k-1}{m-k}\rangle\}$$

and $\mathcal{T}_0 := \{\langle \top \sqsubseteq V_0 \geq \frac{m-k-1}{m-k}\rangle\}$. Every model $\mathcal{I}$ of $\bigcup_{i=0}^{m} \mathcal{T}_i$ and $\delta \in \Delta^\mathcal{I}$ satisfies that $V_0^\mathcal{I}(\delta) \geq \frac{m-k-1}{m-k}$ and $V_i^\mathcal{I}(\delta) \in \{\frac{m-k-1}{m-k}, 1\}$ for $1 \leq i \leq m$. We now define

$$\begin{aligned}\mathcal{T}_\mathcal{V} := \bigcup_{i=0}^{m} \mathcal{T}_i \ \cup\ &\{\langle A \sqsubseteq V_0^{m-k-1} \geq 1\rangle, \langle V_1 \sqcap \ldots \sqcap V_m \sqsubseteq B \geq 1\rangle\} \cup \\ &\{\langle V_0 \sqsubseteq V_{j_1} \sqcap V_{j_2} \geq 1\rangle \mid \{v_{j_1}, v_{j_2}\} \in \mathcal{E}\}.\end{aligned} \tag{1}$$

Theorem 6. *There is a vertex cover of $V, \mathcal{E}$ of size k iff A is not 1-subsumed by B w.r.t. $\mathcal{T}_\mathcal{V}$.*

Proof. Let $S = \{v_{i1}, \ldots, v_{ik}\}$ be a vertex cover of size k. Build the interpretation $\mathcal{I}_S := (\{\delta\}, \cdot^{\mathcal{I}_S})$ with $A^{\mathcal{I}_S}(\delta) := 1/m-k$, $B^{\mathcal{I}_S}(\delta) := 0$, $V_0^{\mathcal{I}_S}(\delta) := \frac{m-k-1}{m-k}$, and for $i, 1 \leq i \leq m$,

$$V_i^{\mathcal{I}_S}(\delta) := \begin{cases} 1 & \text{if } v_i \in S \\ \frac{m-k-1}{m-k} & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mathcal{I}_S$ is a model of $\mathcal{T}_\mathcal{V}$ and $A^{\mathcal{I}_S}(\delta) \Rightarrow B^{\mathcal{I}_S}(\delta) = \frac{m-k-1}{m-k} < 1$.

Conversely, let $\mathcal{I}$ be a model of $\mathcal{T}_\mathcal{V}$ and $\delta \in \Delta^\mathcal{I}$ with $A^\mathcal{I}(\delta) > B^\mathcal{I}(\delta)$. In particular, $A^\mathcal{I}(\delta) \leq 1/m-k$, since otherwise, $B^\mathcal{I}(\delta) = 1$. We can now define $S_\mathcal{I} := \{v_i \mid V_i^\mathcal{I}(\delta) = 1, 1 \leq i \leq m\}$. Since $V_1^\mathcal{I}(\delta) \otimes \ldots \otimes V_m^\mathcal{I}(\delta) < 1/m-k$, there must be at least $m - k$ concept names V_j such that $V_j^\mathcal{I}(\delta) = \frac{m-k-1}{m-k}$, and hence $S_\mathcal{I}$ has at most k elements. Moreover, since $\mathcal{I}$ satisfies the axioms in (1), for every $\{v_{j_1}, v_{j_2}\} \in \mathcal{E}$, at least one of $V_{j_1}^\mathcal{I}(\delta), V_{j_2}^\mathcal{I}(\delta)$ is 1. Thus, $S_\mathcal{I}$ is a vertex cover. □

Corollary 7. *1-subsumption in Ł-$\mathcal{EL}$ is co-*NP*-hard.*

Since $\mathcal{T}_\mathcal{V}$ does not use any roles, hardness holds already in the sublogic of Ł-$\mathcal{EL}$ without roles. We can extend this result with the help of the following theorem.

Theorem 8 ([9]). *Let $\otimes_1, \otimes_2$ be continuous t-norms, $b \in (0, 1)$, and $\otimes$ be the ordinal sum of $((0, b), \otimes_1)$, $((b, 1), \otimes_2)$. Then p-subsumption in $\otimes$-$\mathcal{EL}$ is at least as hard as p-subsumption in $\otimes_2$-$\mathcal{EL}$.*

A direct consequence of this theorem is that 1-subsumption is co-NP-hard in $\otimes$-$\mathcal{EL}$, for any continuous t-norm $\otimes$ that *ends with* the Łukasiewicz t-norm. Using similar reductions to the vertex cover problem, it was previously shown that other subsumption problems are intractable for t-norms that start with Łukasiewicz. The proofs are similar to the one of Theorem 6.

Proposition 9 ([9]). *If $\otimes$ starts with Łukasiewicz, then positive subsumption and p-subsumption in $\otimes$-$\mathcal{EL}$ are co-*NP*-hard.*

Every t-norm that contains the Łukasiewicz t-norm can be expressed as the ordinal sum of two components $((0,b),\otimes_1)$, $((b,1),\otimes_2)$, where $\otimes_2$ starts with Łukasiewicz. Thus, Proposition 9 and Theorem 8 yield the following.

Corollary 10. *If $\otimes$ contains the Łukasiewicz t-norm, then p-subsumption in $\otimes$-$\mathcal{EL}$ is co-*NP*-hard.*

This shows that the best subsumption degree in $\otimes$-$\mathcal{EL}$ cannot be computed in polynomial time if $\otimes$ contains the Łukasiewicz t-norm (unless P = NP).

For positive subsumption there is also a matching tractability result: if the underlying t-norm $\otimes$ does not start with the Łukasiewicz t-norm, then positive subsumption is decidable in polynomial time, as in the crisp case [1,10]. This can be shown by a reduction similar to the one from [5], where consistency in expressive fuzzy DLs is reduced to the corresponding crisp DLs. This reduction transforms a $\otimes$-$\mathcal{EL}$-TBox $\mathcal{T}$ into the crisp TBox

$$\mathcal{T}^{>0} := \{\langle C \sqsubseteq D \geq 1\rangle \mid \langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}, q > 0\}$$

that describes all positive subsumption relations.

Theorem 11 ([9]). *Let $\mathcal{T}$ be a TBox and C_0, D_0 two concepts. Then C_0 is positively subsumed by D_0 w.r.t. $\mathcal{T}$ iff for every crisp model $\mathcal{J}$ of $\mathcal{T}^{>0}$ and $\delta \in \Delta^{\mathcal{J}}$ it holds that $C_0^{\mathcal{J}}(\delta) \leq D_0^{\mathcal{J}}(\delta)$.*

The latter condition in this theorem is equivalent to subsumption between C_0 and D_0 in classical $\mathcal{EL}$, which can be decided in polynomial time [10].

Corollary 12. *If $\otimes$ does not start with Łukasiewicz, then positive subsumption in $\otimes$-$\mathcal{EL}$ is decidable in polynomial time.*

4 A Completion Algorithm for 1-Subsumption

We now develop a completion algorithm in the style of [1,16] that allows us to decide 1-subsumption under the following restrictions. As in Corollary 12, the underlying t-norm $\otimes$ must not start with Łukasiewicz. Furthermore, all roles are restricted to be crisp, i.e. they are always interpreted by fuzzy binary relations using only the values 0 and 1. The third and last restriction is that the underlying TBox $\mathcal{T}$ is restricted to be *normalized*, i.e. all GCIs in $\mathcal{T}$ are of the form

$$\langle A_1 \sqcap A_2 \sqsubseteq B \geq p\rangle,\ \langle A \sqsubseteq \exists r.B \geq p\rangle,\ \langle \exists r.A \sqsubseteq B \geq p\rangle$$

for $A_1, A_1, A, B \in \mathsf{N}_\mathsf{C}^\top := \mathsf{N}_\mathsf{C} \cup \{\top\}$ and $p \in [0,1]$.[4] Contrary to the classical case, $\otimes$-$\mathcal{EL}$-TBoxes cannot be transformed into equivalent normalized ones in general; hence, this restriction does affect the expressivity of the logic.

[4] Notice that $\langle A \sqsubseteq B \geq p\rangle$ is equivalent to $\langle \top \sqcap A \sqsubseteq B \geq p\rangle$.

(CR1) If $q_1 \otimes x_A^n \in S(A, B_1)$, $q_2 \otimes x_A^m \in S(A, B_2)$, and $\langle B_1 \sqcap B_2 \sqsubseteq C \geq p \rangle \in \mathcal{T}$, then add $(p \otimes q_1 \otimes q_2) \otimes x_A^{n+m}$ to $S(A, C)$.

(CR2) If $q \otimes x_A^n \in S(A, B)$ and $\langle B \sqsubseteq \exists r.C \geq p \rangle \in \mathcal{T}$, then add $(p \otimes q) \otimes x_A^n$ to $R(A, r, C)$.

(CR3) If $q_1 \otimes x_A^n \in R(A, r, B)$, $q_2 \otimes x_B^m \in S(B, C)$, and $\langle \exists r.C \sqsubseteq D \geq p \rangle \in \mathcal{T}$, then add $(p \otimes q_1^m \otimes q_2) \otimes x_A^{nm}$ to $S(A, D)$.

Fig. 1. The completion rules

Given such a TBox $\mathcal{T}$, we compute for every $A, B \in \mathsf{N}_\mathsf{C}^\top$, and $r \in \mathsf{N}_\mathsf{R}$ sets $S(A, B)$ and $R(A, r, B)$ containing monomials of the form $q \otimes x_A^n$, where x_A is a variable, $n \geq 0$ is a natural number, and $q \in [0, 1]$. The idea is that, whenever the value of A is $p \in [0, 1]$, then $q \otimes x_A^n \in S(A, B)$ implies that the value of B is at least $q \otimes p^n$, and thus A^n is q-subsumed by B. Similarly, if $q \otimes x_A^n \in R(A, r, B)$, then the value of $\exists r.B$ is greater or equal $q \otimes p^n$. In this way, $S(A, B)$ (or $R(A, r, B)$) describes subsumption relationships between (powers of) A and B (or $\exists r.B$).

We define an order $\succeq$ on such monomials as follows. Given $p, q \in [0, 1]$ and $n, m \geq 0$, we define $q \otimes x^n \succeq p \otimes x^m$ iff $n \leq m$ and $q \geq p$. Note that $q \otimes x^n \succeq p \otimes x^m$ implies that the value of the first monomial for $x \in [0, 1]$ is always greater or equal that of the second monomial. Since these monomials represent lower bounds for the best subsumption degree, it is clear that we only need to add a monomial to $S(A, B)$ or $R(A, r, B)$ if this set does not already contain a larger one. We also never add the trivial monomial 0.

We initialize these sets as $S(A, A) := \{x_A\}$, and $S(A, \top) := S(\top, \top) := \{1\}$ for all $A \in \mathsf{N}_\mathsf{C}$. All other sets $S(A, B)$ and $R(A, r, B)$ are initially empty. We then exhaustively apply the rules from Figure 1. As mentioned before, a monomial is only added to a set if it does not already contain a larger monomial w.r.t. $\succeq$.

The completion rules in Figure 1 generalize those for classical $\mathcal{EL}$ [10] and for G-$\mathcal{EL}$ [16]. The difference to the rules for the Gödel t-norm are caused by the existence of non-idempotent elements in general t-norms. For the Gödel t-norm, the subsumption degree of A^n by B is independent of n, and thus only monomials of the form q or $q \otimes x_A$, i.e. constants or linear terms, can occur in $S(A, B)$.

Note that the sets $S(\top, B)$ for $B \in \mathsf{N}_\mathsf{C}^\top$ can only contain constants, which is why we will often treat $S(\top, B)$ as a value from $[0, 1]$, which is 0 if the set is empty. Furthermore, it is easy to show that any constant added to $S(\top, B)$ is also added to every $S(A, B)$ for $A \in \mathsf{N}_\mathsf{C}^\top$, and vice versa, by applying the same rules with different left-hand sides. Similar arguments apply to $R(\top, r, B)$.

We now argue that the algorithm described above terminates. Consider any $A, B \in \mathsf{N}_\mathsf{C}^\top$. If at some point during the run of the algorithm a monomial $q \otimes x_A^n$ is added to $S(A, B)$ by a rule application, then q must be of the form $p_1 \otimes \ldots \otimes p_m$ for values $p_1, \ldots, p_m$ occurring in $\mathcal{T}$. Once $S(A, B)$ contains $q \otimes x_A^n$, only monomials of the form $q' \otimes x_A^m$, where either $q' > q$ or $m < n$, can be added to $S(A, B)$. Since q' also has to be a combination of values occurring in $\mathcal{T}$, there are only finitely many values q' that satisfy the first condition and are contained in the same component of $\otimes$ as q. Obviously, there are also only finitely many numbers m satisfying the second condition. Furthermore, for each q' there can only be one m

such that $q' \otimes x_A^m \in S(A,B)$, and once there is such an m, it can only be decreased by the following rule applications. Similarly, for each m there can only be one q' with this property, and this q' can only be increased. As mentioned before, there are only finitely many possibilities for q' inside the same component, and once a new q' has been computed that lies in another component, there are again only finitely many possible values exceeding q' in the same component. Since from the values in $\mathcal{T}$ one can only compute values in finitely many components of $\otimes$, this shows that the algorithm can add only finitely many elements to $S(A,B)$ (or $R(A,r,B)$), and hence it always terminates.

Lemma 13. *Let $A, B \in \mathsf{N}_\mathsf{C}^\top$, $r \in \mathsf{N}_\mathsf{R}$, $\mathcal{I}$ be a model of $\mathcal{T}$, and $\delta \in \Delta^\mathcal{I}$.*

- *If $q \otimes x_A^n \in S(A,B)$ and $A^\mathcal{I}(\delta) > 0$, then $q \otimes (A^\mathcal{I}(\delta))^n \leq B^\mathcal{I}(\delta)$.*
- *If $q \otimes x_A^n \in R(A,r,B)$ and $A^\mathcal{I}(\delta) > 0$, then $q \otimes (A^\mathcal{I}(\delta))^n \leq (\exists r.B)^\mathcal{I}(\delta)$.*

Proof. The claim is obviously true after initializing S and R. Assume that it holds after applying several rules and consider the next rule that is applied.

In the case of (**CR1**), consider $q_1 \otimes x_A^n \in S(A,B_1)$, $q_2 \otimes x_A^m \in S(A,B_2)$, $\langle B_1 \sqcap B_2 \sqsubseteq C \geq p \rangle \in \mathcal{T}$, and $A^\mathcal{I}(\delta) > 0$. We thus have $q_1 \otimes (A^\mathcal{I}(\delta))^n \leq B_1^\mathcal{I}(\delta)$, $q_2 \otimes (A^\mathcal{I}(\delta))^m \leq B_2^\mathcal{I}(\delta)$, and $p \otimes B_1^\mathcal{I}(\delta) \otimes B_2^\mathcal{I}(\delta) \leq C^\mathcal{I}(\delta)$. It follows that

$$p \otimes q_1 \otimes q_2 \otimes (A^\mathcal{I}(\delta))^{n+m} \leq p \otimes B_1^\mathcal{I}(\delta) \otimes B_2^\mathcal{I}(\delta) \leq C^\mathcal{I}(\delta),$$

and thus we can add $(p \otimes q_1 \otimes q_2) \otimes x_A^{n+m}$ to $S(A,C)$ without violating the claim.

For (**CR2**), let $q \otimes x_A^n \in S(A,B)$, $\langle B \sqsubseteq \exists r.C \geq p \rangle \in \mathcal{T}$, and $A^\mathcal{I}(\delta) > 0$. By assumption, we have $q \otimes (A^\mathcal{I}(\delta))^n \leq B^\mathcal{I}(\delta)$ and $p \otimes B^\mathcal{I}(\delta) \leq (\exists r.C)^\mathcal{I}(\delta)$, and thus $p \otimes q \otimes (A^\mathcal{I}(\delta))^n \leq (\exists r.C)^\mathcal{I}(\delta)$ as required.

Finally, for the case of (**CR3**), let $q_1 \otimes x_A^n \in R(A,r,B)$, $q_2 \otimes x_B^m \in S(B,C)$, $\langle \exists r.C \sqsubseteq D \geq p \rangle \in \mathcal{T}$, and $A^\mathcal{I}(\delta) > 0$, which yields $q_1 \otimes (A^\mathcal{I}(\delta))^n \leq (\exists r.B)^\mathcal{I}(\delta)$. We first consider the case that $m = 0$. Since $q_1 > 0$ and $\otimes$ does not start with Łukasiewicz, we have $(\exists r.B)^\mathcal{I}(\delta) > 0$. Thus, there is a $\gamma \in \Delta^\mathcal{I}$ with $r^\mathcal{I}(\delta,\gamma) = 1$ and $B^\mathcal{I}(\gamma) > 0$. The assumption yields that $q_2 \leq C^\mathcal{I}(\gamma)$, and thus

$$p \otimes q_1^0 \otimes q_2 \otimes (A^\mathcal{I}(\delta))^0 = p \otimes q_2 \leq p \otimes r^\mathcal{I}(\delta,\gamma) \otimes C^\mathcal{I}(\gamma) \leq p \otimes (\exists r.C)^\mathcal{I}(\delta) \leq D^\mathcal{I}(\delta).$$

For the case of $m \geq 1$, since r is crisp we get

$$\begin{aligned} q_2 \otimes \big((\exists r.B)^\mathcal{I}(\delta)\big)^m &= q_2 \otimes \big(\sup_{\gamma \in \Delta^\mathcal{I}} r^\mathcal{I}(\delta,\gamma) \otimes B^\mathcal{I}(\gamma)\big)^m \\ &= q_2 \otimes \sup_{\substack{\gamma \in \Delta^\mathcal{I} \\ B^\mathcal{I}(\gamma) > 0}} r^\mathcal{I}(\delta,\gamma) \otimes (B^\mathcal{I}(\gamma))^m \leq \sup_{\substack{\gamma \in \Delta^\mathcal{I} \\ B^\mathcal{I}(\gamma) > 0}} r^\mathcal{I}(\delta,\gamma) \otimes C^\mathcal{I}(\gamma) \leq (\exists r.C)^\mathcal{I}(\delta). \end{aligned}$$

This implies that

$$p \otimes q_1^m \otimes q_2 \otimes (A^\mathcal{I}(\delta))^{nm} \leq p \otimes q_2 \otimes \big((\exists r.B)^\mathcal{I}(\delta)\big)^m \leq p \otimes (\exists r.C)^\mathcal{I}(\delta) \leq D^\mathcal{I}(\delta).$$

Hence, the claim is still satisfied after adding $(p \otimes q_1^m \otimes q_2) \otimes x_A^{nm}$ to $S(A,D)$. ☐

We now show that this algorithm is complete for deciding 1-subsumptions.

Lemma 14. *For every $A, B \in \mathsf{N}_\mathsf{C}^\top$ with $A \sqsubseteq_\mathcal{T}^1 B$ and all $p \in [0,1]$, it holds that*

$$p \leq \max_{q \otimes x_A^n \in S(A,B)} q \otimes p^n.$$

Proof. We construct a canonical model $\mathcal{I}$ of $\mathcal{T}$ from which we can read off all 1-subsumptions. Its domain is $\Delta^\mathcal{I} := \{A_p \mid A \in \mathsf{N}_\mathsf{C}^\top,\ p \in [0,1]\}$. Given $C \in \mathsf{N}_\mathsf{C}$, $r \in \mathsf{N}_\mathsf{R}$, $A, B \in \mathsf{N}_\mathsf{C}^\top$, and $p, p' \in [0,1]$, we set $C^\mathcal{I}(A_p) := \max_{q \otimes x_A^n \in S(A,C)} q \otimes p^n$, where the empty maximum is 0, and

$$r^\mathcal{I}(A_p, B_{p'}) := \begin{cases} 1 & \text{if } p' = \max_{q \otimes x_A^n \in R(A,r,B)} q \otimes p^n, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that it also holds that $\top^\mathcal{I}(A_p) = \max_{q \otimes x_A^n \in S(A,\top)} q \otimes p^n$ since $S(A,\top)$ is always $\{1\}$. Furthermore, for any $A \in \mathsf{N}_\mathsf{C}^\top$ and $p \in [0,1]$ we have

$$A^\mathcal{I}(A_p) = \max_{q \otimes x_A^n \in S(A,A)} q \otimes p^n = \max\{S(\top, A), p\}.$$

To show that $\mathcal{I}$ is actually a model of $\mathcal{T}$, consider first an axiom of the form $\langle B_1 \sqcap B_2 \sqsubseteq C \geq p \rangle$ in $\mathcal{T}$ and a domain element $A_{p'} \in \Delta^\mathcal{I}$. By (**CR1**), we have

$$\begin{aligned} p \otimes B_1^\mathcal{I}(A_{p'}) \otimes B_2^\mathcal{I}(A_{p'}) &= \max_{q_1 \otimes x_A^n \in S(A,B_1)} \max_{q_2 \otimes x_A^m \in S(A,B_2)} p \otimes q_1 \otimes q_2 \otimes (p')^{n+m} \\ &\leq \max_{q \otimes x_A^n \in S(A,C)} q \otimes (p')^n = C^\mathcal{I}(A_{p'}). \end{aligned}$$

For an axiom $\langle B \sqsubseteq \exists r.C \geq p \rangle \in \mathcal{T}$, let $p'' := \max_{q \otimes x_A^n \in R(A,r,C)} q \otimes (p')^n$. We get

$$\begin{aligned} p \otimes B^\mathcal{I}(A_{p'}) &= \max_{q \otimes x_A^n \in S(A,B)} p \otimes q \otimes (p')^n \leq \max_{q \otimes x_A^n \in R(A,r,C)} q \otimes (p')^n = p'' \\ &\leq \max\{S(\top, C), p''\} = C^\mathcal{I}(C_{p''}) = r^\mathcal{I}(A_{p'}, C_{p''}) \otimes C^\mathcal{I}(C_{p''}) \\ &\leq \sup_{D_{p'''} \in \Delta^\mathcal{I}} r^\mathcal{I}(A_{p'}, D_{p'''}) \otimes C^\mathcal{I}(D_{p'''}) = (\exists r.C)^\mathcal{I}(A_{p'}). \end{aligned}$$

Finally, for an axiom $\langle \exists r.C \sqsubseteq D \rangle \in \mathcal{T}$, let $p_B := \max_{q_1 \otimes x_A^n \in R(A,r,B)} q_1 \otimes (p')^n$ for every $B \in \mathsf{N}_\mathsf{C}^\top$. By (**CR3**), we have

$$\begin{aligned} p \otimes (\exists r.C)^\mathcal{I}(A_{p'}) &= \sup_{B_{p''} \in \Delta^\mathcal{I}} p \otimes r^\mathcal{I}(A_{p'}, B_{p''}) \otimes C^\mathcal{I}(B_{p''}) = \max_{B \in \mathsf{N}_\mathsf{C}^\top} p \otimes C^\mathcal{I}(B_{p_B}) \\ &= \max_{B \in \mathsf{N}_\mathsf{C}^\top} \max_{q_2 \otimes x_B^m \in S(B,C)} p \otimes q_2 \otimes p_B^m \\ &= \max_{B \in \mathsf{N}_\mathsf{C}^\top} \max_{q_1 \otimes x_A^n \in R(A,r,B)} \max_{q_2 \otimes x_B^m \in S(B,C)} p \otimes q_2 \otimes q_1^m \otimes (p')^{nm} \\ &\leq \max_{B \in \mathsf{N}_\mathsf{C}^\top} \max_{q \otimes x_A^n \in S(A,D)} q \otimes (p')^n = D^\mathcal{I}(A_{p'}). \end{aligned}$$

Consider now $A, B \in \mathsf{N}_\mathsf{C}^\top$ with $A \sqsubseteq_\mathcal{T}^1 B$, and any $p \in [0,1]$. Then we have $p \leq \max\{S(\top, A), p\} = A^\mathcal{I}(A_p) \leq B^\mathcal{I}(A_p) = \max_{q \otimes x_A^n \in S(A,B)} q \otimes p^n$. □

We now show how to employ the algorithm to decide 1-subsumptions between concept names in $\otimes$-$\mathcal{EL}$. The actual decision procedure depends on the structure of $\otimes$. More precisely, we consider the smallest $b \in [0,1]$ such that all elements in $[b,1]$ are idempotent w.r.t. $\otimes$. This means that $\otimes$ is isomorphic to the Gödel t-norm on $[b,1]$, or equivalently, that the representation of $\otimes$ according to Theorem 1 has no component overlapping $[b,1]$. Since $\otimes$ is fixed, we assume in the following that b is known or easily computable from the representation of $\otimes$.

Theorem 15. *Let $A, B \in \mathsf{N}_\mathsf{C}^\top$. Then $A \sqsubseteq_\mathcal{T}^1 B$ iff either (i) $\{x_A, 1\} \cap S(A,B) \neq \emptyset$, or (ii) $\{q, x_A^n\} \subseteq S(A,B)$ for $q \geq b$ and $n \geq 2$.*

Proof. **[if]** Let $\mathcal{I}$ be a model of $\mathcal{T}$ and $\delta \in \Delta^\mathcal{I}$. We show that $A^\mathcal{I}(\delta) \leq B^\mathcal{I}(\delta)$. If $A^\mathcal{I}(\delta) = 0$, then this obviously holds. If $A^\mathcal{I}(\delta) > 0$, then Lemma 13 yields $A^\mathcal{I}(\delta) \leq B^\mathcal{I}(\delta)$, $A^\mathcal{I}(\delta) \leq 1 \leq B^\mathcal{I}(\delta)$, or $q \leq B^\mathcal{I}(\delta)$ and $(A^\mathcal{I}(\delta))^n \leq B^\mathcal{I}(\delta)$, depending on $S(A,B)$. In the last case, we have either $A^\mathcal{I}(\delta) < b \leq B^\mathcal{I}(\delta)$, or $A^\mathcal{I}(\delta) \geq b$ and then $A^\mathcal{I}(\delta) = (A^\mathcal{I}(\delta))^n \leq B^\mathcal{I}(\delta)$.
[only if] Assume first that $S(A,B)$ contains a constant q with $b \leq q < 1$. In this case, every monomial in $S(A,B)$ must be of the form $q' \otimes x_A^n$ with $q' < 1$. For all these monomials, it holds that $q' \otimes q^n = q' \otimes q < q$. By Lemma 14, this implies $A \not\sqsubseteq_\mathcal{T}^1 B$. Otherwise, if $S(A,B)$ contains a constant q, then it must satisfy $q < b$. For all monomials $q' \otimes x_A^n \in S(A,B)$ it then holds that $q' < 1$ or $n \geq 2$. If $q' < 1$, then we have $q' \otimes p^n \leq q' \otimes p < p$ for all $p \in (0,1]$. If $n \geq 2$, then $q' \otimes p^n \leq p^n < p$ holds for all idempotent elements $p \in (0,b)$. Thus, we have $p > \max_{q' \otimes x_A^n \in S(A,B)} q' \otimes p^n$ for all $p \in (q,b)$, where we set $q := 0$ if $S(A,B)$ does not contain any constant. Again, Lemma 14 yields $A \not\sqsubseteq_\mathcal{T}^1 B$. □

For t-norms with $b = 1$, this means that we can restrict the completion algorithm to consider only 1 and x_A for the sets $S(A,B)$. Once a smaller constant or a larger exponent for x_A is introduced, it can never lead to another entry of the form 1 or x_A, and is thus not necessary to decide 1-subsumption. A special case is the rule (**CR3**) for $m = 0$, since then also a smaller monomial in $R(A,r,B)$ can cause 1 to be added to $S(A,D)$. However, this does not depend on the actual monomial in $R(A,r,B)$, but only on its existence. Since entries in $R(A,r,B)$ can only be produced by (**CR2**), retaining the information whether $S(A,B)$ or $R(A,r,B)$ contain some non-zero monomial is sufficient. As there are only polynomially many sets $S(A,B)$ and $R(A,r,B)$, and for each set we need to retain 3 bits of information, 1-subsumptions can be decided in polynomial time if $b = 1$.

For t-norms with $b < 1$, deciding 1-subsumption additionally depends on the constants in $S(A,B)$. However, as above, we can compute all constants for $S(A,B)$ and $R(A,r,B)$ while only retaining those constants and the information whether the sets contain a non-constant monomial. Furthermore, we can stop the computation of larger constants for $S(A,B)$ once we have exceeded b. Once we have computed these constants, we can proceed as follows. For the sets $S(A,B)$ containing no constant greater or equal b, we simply have to decide whether they contain 1 or x_A as above. For the other sets, the exponents of the monomials $q' \otimes x_A^n$ are irrelevant since either the value of A is below b, and thus below the value of B, or the value of A is above b, and then multiplying it with itself does

Table 2. A summary of the complexity results

	positive subs.	p-subs.	1-subs.	1-subs. w.r.t. crisp roles, normalized TBoxes
in PTIME	not $((0,b)\otimes_Ł)$	—	—	not $((0,b),\otimes_Ł)$
co-NP-hard	$((0,b),\otimes_Ł)$	$((a,b),\otimes_Ł)$	$((a,1),\otimes_Ł)$	—

not change it. Thus, we can apply (**CR1**)–(**CR3**) while treating all non-zero exponents n as 1. Since again it suffices to restrict to those monomials $q' \otimes x_A$ with $q' = 1$, 1-subsumptions can also be decided in polynomial time if $b < 1$.

Corollary 16. *If $\otimes$ does not start with Łukasiewicz, then* 1*-subsumption between concept names in $\otimes$-$\mathcal{EL}$ w.r.t. normalized TBoxes and crisp roles is decidable in polynomial time.*

Consider in particular any t-norm $\otimes$ that ends with (but does not start with) the Łukasiewicz t-norm. From Corollary 16, we know that 1-subsumption of concept names in $\otimes$-$\mathcal{EL}$ is decidable in polynomial time, if the TBox is normalized, and reasoning is restricted to crisp roles. On the other hand, by Corollary 7 and Theorem 8, we know that 1-subsumption w.r.t. general TBoxes is co-NP-hard in this logic. Moreover, the constructions used for these results do not use any roles, and hence the restriction to crisp roles does not affect the hardness. This means that general TBoxes are strictly more expressive than normalized ones.

5 Conclusions

We have analyzed subsumption problems in fuzzy $\mathcal{EL}$ with t-norm semantics. For the complexity of deciding positive subsumption, there is a dichotomy between co-NP-hard for t-norms that start with Łukasiewicz and polynomial for t-norms that do not. For the latter case, positive subsumption is linearly reducible to subsumption in classical $\mathcal{EL}$. This dichotomy is akin the complexity of deciding TBox consistency in expressive fuzzy DLs: for t-norms starting with Łukasiewicz, the problem is undecidable [6,7,11], but linearly reducible to classical reasoning for all other t-norms [4,5].

Deciding p-subsumption exhibits a different complexity pattern. There, the co-NP lower bound holds for any t-norm containing Łukasiewicz. We have not been able to obtain complexity results for other t-norms, beyond the previously known case of the Gödel t-norm. For 1-subsumption we have shown intractability for any t-norm ending with Łukasiewicz. These results are summarized in Table 2.

We have also presented a completion algorithm for deciding 1-subsumption w.r.t. normalized TBoxes, if the semantics is restricted to crisp roles and the t-norm does not start with Łukasiewicz. This is only a first step towards an algorithm capable of deciding p-subsumption in general. Due to our hardness results, we cannot expect to find a polynomial-time algorithm capable of classifying TBoxes that are not in normal form. As future work, we plan to further understand the cases where reasoning becomes intractable, and develop algorithms that match the theoretical complexity of these problems.

References

1. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ envelope. In: Kaelbling, L.P., Saffiotti, A. (eds.) Proc. of the 19th Int. Joint Conf. on Artificial Intelligence (IJCAI'05). pp. 364–369. Professional Book Center (2005)
2. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F. (eds.): The Description Logic Handbook: Theory, Implementation, and Applications. Cambridge University Press, 2nd edn. (2007)
3. Bobillo, F., Straccia, U.: Finite fuzzy description logics and crisp representations. In: Bobillo, F., da Costa, P.C.G., d'Amato, C., Fanizzi, N., Laskey, K., Laskey, K., Lukasiewicz, T., Nickles, M., Pool, M. (eds.) Uncertainty Reasoning for the Semantic Web II, Lecture Notes in Computer Science, vol. 7123, pp. 102–121. Springer-Verlag (2013)
4. Borgwardt, S., Distel, F., Peñaloza, R.: Gödel negation makes unwitnessed consistency crisp. In: Kazakov, Y., Lembo, D., Wolter, F. (eds.) Proc. of the 25th Int. Workshop on Description Logics (DL'12). CEUR Workshop Proceedings, vol. 846, pp. 103–113 (2012)
5. Borgwardt, S., Distel, F., Peñaloza, R.: How fuzzy is my fuzzy description logic? In: Gramlich, B., Miller, D., Sattler, U. (eds.) Proc. of the 6th Int. Joint Conf. on Automated Reasoning (IJCAR'12). Lecture Notes in Artificial Intelligence, vol. 7364, pp. 82–96. Springer-Verlag (2012)
6. Borgwardt, S., Peñaloza, R.: Non Gödel negation makes unwitnessed consistency undecidable. In: Kazakov, Y., Lembo, D., Wolter, F. (eds.) Proc. of the 25th Int. Workshop on Description Logics (DL'12). CEUR Workshop Proceedings, vol. 846, pp. 411–421 (2012)
7. Borgwardt, S., Peñaloza, R.: Undecidability of fuzzy description logics. In: Brewka, G., Eiter, T., McIlraith, S.A. (eds.) Proc. of the 13th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'12). pp. 232–242. AAAI Press (2012)
8. Borgwardt, S., Peñaloza, R.: The complexity of lattice-based fuzzy description logics. Journal on Data Semantics 2(1), 1–19 (2013)
9. Borgwardt, S., Peñaloza, R.: Positive subsumption in fuzzy $\mathcal{EL}$ with general t-norms. In: Proc. of the 23rd Int. Joint Conf. on Artificial Intelligence (IJCAI'13). AAAI Press (2013), to appear.
10. Brandt, S.: Polynomial time reasoning in a description logic with existential restrictions, GCI axioms, and - what else? In: de Mántaras, R.L., Saitta, L. (eds.) Proc. of the 16th Eur. Conf. on Artificial Intelligence (ECAI'04). pp. 298–302. IOS Press (2004)
11. Cerami, M., Straccia, U.: On the (un)decidability of fuzzy description logics under Łukasiewicz t-norm. Information Sciences 227, 1–21 (2013)
12. Hájek, P.: Metamathematics of Fuzzy Logic (Trends in Logic). Springer-Verlag (2001)
13. Hájek, P.: Making fuzzy description logic more general. Fuzzy Sets and Systems 154(1), 1–15 (2005)
14. Karp, R.: Reducibility among combinatorial problems. In: Miller, R.E., Thatcher, J.W. (eds.) Proc. of a Symp. on the Complexity of Computer Computations, pp. 85–103. Plenum Press (1972)
15. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Trends in Logic, Studia Logica Library, Springer-Verlag (2000)

16. Mailis, T., Stoilos, G., Simou, N., Stamou, G., Kollias, S.: Tractable reasoning with vague knowledge using fuzzy $\mathcal{EL}^{++}$. Journal of Intelligent Information Systems 39, 399–440 (2012)
17. Mostert, P.S., Shields, A.L.: On the structure of semigroups on a compact manifold with boundary. Annals of Mathematics 65(1), 117–143 (1957)
18. Straccia, U.: Reasoning within fuzzy description logics. Journal of Artificial Intelligence Research 14, 137–166 (2001)
19. Tresp, C.B., Molitor, R.: A description logic for vague knowledge. In: Prade, H. (ed.) Proc. of the 13th Eur. Conf. on Artificial Intelligence (ECAI'98). pp. 361–365. John Wiley and Sons (1998)
20. Zhou, Z., Qi, G., Liu, C., Hitzler, P., Mutharaju, R.: Reasoning with fuzzy-$\mathcal{EL}^{+}$ ontologies using MapReduce. In: Raedt, L.D., Bessiere, C., Dubois, D., Doherty, P., Frasconi, P., Heintz, F., Lucas, P. (eds.) Proc. of the 20th Eur. Conf. on Artificial Intelligence (ECAI'12). Frontiers in Artificial Intelligence and Applications, vol. 242, pp. 933–934. IOS Press (2012)