On the Expressiveness of TPTL and MTL over ω-Data Words

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Metric Temporal Logic (MTL) and Timed Propositional Temporal Logic (TPTL) are prominent extensions of Linear Temporal Logic to specify properties about data languages. In this paper, we consider the class of data languages of non-monotonic data words over the natural numbers. We prove that, in this setting, TPTL is strictly more expressive than MTL. To this end, we introduce Ehrenfeucht-Fraïssé (EF) games for MTL. Using EF games for MTL, we also prove that the MTL definability decision problem ("Given a TPTL-formula, is the language defined by this formula definable in MTL?") is undecidable. We also define EF games for TPTL, and we show the effect of various syntactic restrictions on the expressiveness of MTL and TPTL.

1 Introduction

Recently, verification and analysis of sets of *data words* have gained a lot of interest [18, 12, 10, 4, 5, 6, 7]. Here we consider ω-words, *i.e.*, infinite sequences over $\Sigma \times D$, where Σ is a finite set of labels, and D is a potentially infinite set of *data values*. One prominent example of data words are *timed words*, used in the analysis of real-time systems [1]. In this paper, we consider data words as behavioral models of one-counter machines. Therefore, in contrast to timed words, the sequence of data values within the word may be non-monotonic, and we choose the set of natural numbers as data domain. It is straightforward to adapt our results to the data domain of integers. In timed words, intuitively, the sequence of data values describes the timestamps at which the properties from the labels set Σ hold. Non-monotonic sequences of natural numbers, instead, can model the variation of an observed value during a time elapse: we can think of the heartbeat rate recorded by a cardiac monitor, atmospheric pressure, humidity or temperature measurements obtained from a meteorological station. For example, let $\mathsf{Weather} = \{\mathsf{sunny}, \mathsf{cloudy}, \mathsf{rainy}\}$ be a set of labels. A data word modeling the changing of the weather and highest temperature day after day could be:

$$(\mathsf{rainy}, 10)(\mathsf{cloudy}, 8)(\mathsf{sunny}, 12)(\mathsf{sunny}, 13)\ldots$$

For reasoning about data words, we consider extensions of *Linear Temporal Logic* (LTL, for short). One of these extensions is FreezeLTL, which extends LTL with a *freeze quantifier* that stores the current data value in a register variable. One can then check whether in a later position in the data word the data value equals the value stored in the register or not. Model checking one-counter machines with this logic is in general undecidable [12], and so is the satisfiability problem [10]. A good number of recent publications deal with decidable and undecidable fragments of FreezeLTL [10, 11, 12, 13].

Originally, the freeze quantifier was introduced in *Timed Propositional Temporal Logic* (TPTL, for short) [3]. Here, in contrast to FreezeLTL, a data value d can be compared to a register value x using

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linear inequations of the form, *e.g.*, $d - x \leq 2$. Another widely used logic in the context of real-time systems is *Metric Temporal Logic* (MTL, for short) [16]. MTL extends LTL by constraining the temporal operators with intervals over the non-negative reals. It is well known that every MTL-formula can be effectively translated into an equivalent formula in TPTL. For the other direction, however, it turns out that the result depends on the data domain. For *monotonic data words* over the natural numbers, Alur and Henzinger [2] proved that MTL and TPTL are equally expressive. For timed words over the non-negative reals, instead, Bouyer et al. [8] showed that TPTL is strictly more expressive than MTL.

Both logics, however, have not gained much attention in the specification of non-monotonic data words. Recently we studied the decidability and complexity of MTL, TPTL and some of their fragments over non-monotonic data words [9], but still not much is known about their relative expressiveness, albeit they can express many interesting properties. To continue our example, using the MTL-formula $(\mathsf{sunny}\ \mathsf{U}_{[-3,-1]}\ \mathsf{cloudy})$ over the labels set Weather, we can express the following property: it is sunny until it becomes cloudy and the highest temperature has decreased of 1 to 3 degrees. The following TPTL-formula expresses the fact that, at least three days from now, the highest temperature will be the same as today: $x.\mathsf{FFF}(x = 0)$. Over a data word, this formula expresses that there is a point whose data value is the same as that of the present one after at least three points. The main advantage of MTL with respect to TPTL is its concise syntax. It would be practical if we could show that, as in the case of monotonic data words over the natural numbers, MTL equals TPTL on data words. The goal of this paper is to investigate the relative expressiveness of TPTL and MTL when evaluated over data words.

In this paper, we show as a main result that for data words TPTL is strictly more expressive than MTL. More detailed, we use the formula $x.\mathsf{F}(b \wedge \mathsf{F}(c \wedge x \leq 2))$ to separate TPTL and MTL. This is the same formula used in the paper by Bouyer et al. [8] to separate these two logics over timed words. We also show that the simpler TPTL-formula $x.\mathsf{FFF}(x = 0)$ is not definable in MTL. Note that this formula is in the unary fragment of FreezeLTL, which is very restrictive. The intuitive reason for the difference in expressiveness is that, using register variables, we can store data values at any position of a word to compare them with a later position, and it is possible to check that other properties are verified in between. This cannot be done using the constrained temporal operators in MTL. This does not result in a gap in expressiveness in the monotonic data words setting, because the monotonicity of the data sequence does not allow arbitrary values between two positions of a data word.

As a main tool for showing this result, we introduce *quantitative* versions of Ehrenfeucht-Fraïssé (EF) games for MTL and TPTL. In model theory, EF games are mainly used to prove inexpressibility results for first-order logic. Etessami and Wilke [14] introduced the EF game for LTL and used it to show that the Until Hierarchy for LTL is strict. Using our EF games for MTL and TPTL, we prove a number of results concerning the relation between the expressive power of TPTL and MTL, as well as between different fragments of both logics. We investigate the effects of restricting the syntactic resources. For instance, we show that TPTL that permits two register variables is strictly more expressive than TPTL restricted to one register variable. We also use EF games to show that the following problem is undecidable: given a TPTL-formula φ, is there an MTL-formula equivalent to φ?

We remark that quantitative EF games provide a very general and intuitive mean to prove results concerning the expressive power of quantitative logics. We would also like to point out that recently an extension of Etessami and Wilke's EF games has been defined [17] to investigate relative expressiveness of some fragments of the real-time version of MTL over *finite* timed words only. The proof of Theorem 1 in [17] relies on the fact that there is an integer bound on the timestamps of a finite timed word to deal with the potentially infinite number of equivalence classes of MTL formulas. It is not clear how this can be extended to *infinite* timed words. In contrast to this, the results in our paper using EF games can also be applied to *finite* data words.

2 Metric Temporal Logic and Timed Propositional Temporal Logic

In this section, we define two quantitative extensions of LTL: MTL and TPTL. The logics are evaluated over *data words*, defined in the following.

We use $\mathbb{Z}$ and $\mathbb{N}$ to denote the set of integers and the set of non-negative integers, respectively. Let P be a finite set of propositional variables. An *ω-data word*, or simply *data word*, w is an infinite sequence $(P_0,d_0)(P_1,d_1)\ldots$ of pairs in $2^{\mathsf{P}} \times \mathbb{N}$. Let $i \in \mathbb{N}$, we use $w[i]$ to denote the data word $(P_i,d_i)(P_{i+1},d_{i+1})\ldots$ and use $(2^{\mathsf{P}} \times \mathbb{N})^{\omega}$ to denote the set of all data words.

2.1 Metric Temporal Logic

The set of formulas of MTL is built up from P by boolean connectives and a constraining version of the *until* operator:

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathsf{U}_I \varphi_2$$

where $p \in \mathsf{P}$ and $I \subseteq \mathbb{Z}$ is a (half-)open or (half-)closed interval over the integers, possibly unbounded. We use pseudo-arithmetics expressions to denote intervals, *e.g.*, ≥ 1 to denote $[1,+\infty)$. If $I = \mathbb{Z}$, then we may omit the annotation I on U_I.

Formulas in MTL are interpreted over data words. Let $w = (P_0,d_0)(P_1,d_1)\ldots$ be a data word, and let $i \in \mathbb{N}$. We define the *satisfaction relation for* MTL, denoted by $\models_{\mathsf{MTL}}$, inductively as follows:

$$\begin{aligned}
&(w,i) \models_{\mathsf{MTL}} p \text{ iff } p \in P_i, \quad (w,i) \models_{\mathsf{MTL}} \neg\varphi \text{ iff } (w,i) \not\models_{\mathsf{MTL}} \varphi,\\
&(w,i) \models_{\mathsf{MTL}} \varphi_1 \wedge \varphi_2 \text{ iff } (w,i) \models_{\mathsf{MTL}} \varphi_1 \text{ and } (w,i) \models_{\mathsf{MTL}} \varphi_2,\\
&(w,i) \models_{\mathsf{MTL}} \varphi_1 \mathsf{U}_I \varphi_2 \text{ iff } \exists j > i \text{ such that } (w,j) \models_{\mathsf{MTL}} \varphi_2,\, d_j - d_i \in I,\\
&\qquad\qquad \text{and } \forall i < k < j, (w,k) \models_{\mathsf{MTL}} \varphi_1.
\end{aligned}$$

We say that a data word *satisfies* an MTL-formula φ, written $w \models_{\mathsf{MTL}} \varphi$, if $(w,0) \models_{\mathsf{MTL}} \varphi$. We use the following syntactic abbreviations: $\texttt{True} := p \vee \neg p$, $\texttt{False} := \neg\texttt{True}$, $\mathsf{X}_I\varphi := \texttt{False}\mathsf{U}_I\varphi$, $\mathsf{F}_I\varphi := \texttt{True}\mathsf{U}_I\varphi$. Note that the use of the *strict* semantics for the until operator is essential to define the next operator X_I.

Example. The following formula expresses the fact that the weather is sunny until it becomes cloudy and the temperature has decreased from one to three degrees. Furthermore in the future it will rain and the temperature will increase by at least one degree:

$$\mathsf{sunny}\ \mathsf{U}_{[-3,-1]}\,(\mathsf{cloudy} \wedge \mathsf{F}_{\geq 1}\ \mathsf{rainy}). \tag{1}$$

2.2 Timed Propositional Temporal Logic

Given an infinite countable set X of *register variables*, the set of formulas of TPTL is defined as follows:

$$\varphi ::= p \mid x \in I \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathsf{U} \varphi_2 \mid x.\varphi$$

where $p \in \mathsf{P}$, $x \in X$ and I is an interval over the integers, defined as for MTL. We will use pseudo-arithmetic expressions to denote intervals, *e.g.*, $x < 0$ denotes $x \in (0,-\infty)$. Intuitively, $x.\varphi$, means that we are *resetting* x to the current data value, and $x \in I$ means that, compared to the last time that we reset x, the data value has increased or decreased within the margins of the interval I.

Formulas in TPTL are interpreted over data words. A *register valuation* ν is a function from X to $\mathbb{N}$. Let $w = (P_0, d_0)(P_1, d_1)\dots$ be a data word, let ν be a register valuation, and let $i \in \mathbb{N}$. The satisfaction relation for TPTL, denoted by $\models_{\mathsf{TPTL}}$, is inductively defined in a similar way as for MTL; we only give the definitions for the new formulas:

$$(w,i,\nu) \models_{\mathsf{TPTL}} x \in I \text{ iff } d_i - \nu(x) \in I,$$
$$(w,i,\nu) \models_{\mathsf{TPTL}} x.\varphi \text{ iff } (w,i,\nu[x \mapsto d_i]) \models_{\mathsf{TPTL}} \varphi,$$
$$(w,i,\nu) \models_{\mathsf{TPTL}} \varphi_1 \mathsf{U} \varphi_2 \text{ iff } \exists j > i, (w,j,\nu) \models_{\mathsf{TPTL}} \varphi_2, \forall i < k < j, (w,k,\nu) \models_{\mathsf{TPTL}} \varphi_2.$$

Here, $\nu[x \mapsto d_i]$ is the valuation that agrees with ν on all $y \in X \setminus \{x\}$, and maps x to d_i. We say that a data word w satisfies a TPTL-formula φ, written $w \models_{\mathsf{TPTL}} \varphi$, if $(w, 0, \bar{0}) \models_{\mathsf{TPTL}} \varphi$. Here, $\bar{0}$ denotes the valuation that maps each register variable to d_0. We use the same syntactic abbreviations as for MTL where the interval I for the temporal operators is ignored.

In the following, we define some fragments of TPTL. Given $n \geq 1$, we use TPTL^n to denote the set of TPTL-formulas that use at most n different register variables. The *unary fragment of* TPTL, denoted by $\mathsf{UnaTPTL}$, is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid x \in I \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{F}\varphi \mid \mathsf{X}\varphi \mid x.\varphi$$

We define $\mathsf{FreezeLTL}$ to be the subset of TPTL-formulas where the formula '$x \in I$' is restricted to be of the form '$x \in [0,0]$'. We denote combinations of these fragments in the expected manner; *e.g.*, $\mathsf{UnaFreezeLTL}^1$ denotes the unary fragment of TPTL in which only one register variable and equality checks of the form '$x \in [0,0]$' are allowed.

Example. The MTL-formula (1) in the above example is equivalent to the TPTL^1-formula

$$x.[\mathsf{sunny}\ \mathsf{U}\ (x \in [-3,-1] \wedge \mathsf{cloudy} \wedge x.\mathsf{F}\ (x \geq 1 \wedge \mathsf{rainy}))].$$

The formulas $x.((\mathsf{cloudy} \wedge x \leq 2)\mathsf{U}\ \mathsf{sunny})$ and $x.\mathsf{F}\ (\mathsf{cloudy} \wedge \mathsf{F}\ (\mathsf{sunny} \wedge x \leq 2))$, over the labels set $\mathsf{Weather}$ express the following properties:

1. The weather will eventually become sunny. Until then it is cloudy every day and the temperature is at most two degrees higher than the temperature at the present day.
2. It will be cloudy in the future, later it will become sunny, and the temperature will have increased by at most 2 degrees.

2.3 Relative Expressiveness

Let $\mathcal{L}$ and $\mathcal{L}'$ be two logics interpreted over elements in $(2^{\mathsf{P}} \times \mathbb{N})^\omega$, and $\varphi \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$ be two formulas. Define $L(\varphi) = \{w \in (2^{\mathsf{P}} \times \mathbb{N})^\omega \mid w \text{ satisfies } \varphi\}$. We say that φ is *equivalent* to φ' if $L(\varphi) = L(\varphi')$. Given a data language $\mathbf{L} \subseteq (2^{\mathsf{P}} \times \mathbb{N})^\omega$, we say that $\mathbf{L}$ is *definable in* $\mathcal{L}$ if there is a formula $\varphi \in \mathcal{L}$ such that $L(\varphi) = \mathbf{L}$. We say that a formula ψ is definable in $\mathcal{L}$ if $L(\psi)$ is definable in $\mathcal{L}$. We say that $\mathcal{L}'$ is *at least as expressive as* $\mathcal{L}$, written $\mathcal{L} \preccurlyeq \mathcal{L}'$, if each formula of $\mathcal{L}$ is definable in $\mathcal{L}'$. It is *strictly more expressive*, written $\mathcal{L} \prec \mathcal{L}'$ if, additionally, there is a formula in $\mathcal{L}'$ that is not definable in $\mathcal{L}$. Further, $\mathcal{L}$ and $\mathcal{L}'$ are *equally expressive*, written $\mathcal{L} \equiv \mathcal{L}'$, if $\mathcal{L} \preccurlyeq \mathcal{L}'$ and $\mathcal{L}' \preccurlyeq \mathcal{L}$. $\mathcal{L}$ and $\mathcal{L}'$ are *incomparable*, if neither $\mathcal{L} \preccurlyeq \mathcal{L}'$ nor $\mathcal{L}' \preccurlyeq \mathcal{L}$.

In this paper we are interested in the relative expressiveness of (fragments of) MTL and TPTL. It is straightforward to translate an MTL-formula into an equivalent TPTL^1-formula. So it can easily be seen that TPTL^1 is as least as expressive as MTL. However, we will show that there exist some TPTL^1-formulas that are not definable in MTL. For this we introduce the Ehrenfeucht-Fraïssé game for MTL. Before, we define the important notion of *until rank* of a formula.

2.4 Until Rank

The *until rank* of an MTL-formula φ, denoted by $\mathsf{Urk}(\varphi)$, is defined by induction on the structure of the formula:

- $\mathsf{Urk}(p) = 0$ for every $p \in \mathsf{P}$,
- $\mathsf{Urk}(\neg\varphi) = \mathsf{Urk}(\varphi)$, $\mathsf{Urk}(\varphi_1 \wedge \varphi_2) = \max\{\mathsf{Urk}(\varphi_1), \mathsf{Urk}(\varphi_2)\}$, and
- $\mathsf{Urk}(\varphi_1 \mathsf{U}_I \varphi_2) = \max\{\mathsf{Urk}(\varphi_1), \mathsf{Urk}(\varphi_2)\} + 1$.

We use $\mathsf{Cons}(\mathbb{Z})$ to denote the set $\{S \cup \{-\infty, +\infty\} \mid S \subseteq \mathbb{Z}\}$ and $\mathsf{FCons}(\mathbb{Z})$ for the subset of $\mathsf{Cons}(\mathbb{Z})$ which contains all *finite* sets in $\mathsf{Cons}(\mathbb{Z})$. Let $\mathcal{I} \in \mathsf{Cons}(\mathbb{Z})$, $k \in \mathbb{N}$. Define

$$\mathsf{MTL}^{\mathcal{I}} = \{\varphi \in \mathsf{MTL} \mid \text{the endpoints of } I \text{ in each operator } \mathsf{U}_I \text{ in } \varphi \text{ are in } \mathcal{I}\},$$
$$\mathsf{MTL}_k = \{\varphi \in \mathsf{MTL} \mid \mathsf{Urk}(\varphi) \le k\}, \quad \mathsf{MTL}_k^{\mathcal{I}} = \mathsf{MTL}_k \cap \mathsf{MTL}^{\mathcal{I}}.$$

It is easy to check that $\mathsf{MTL} = \bigcup_{\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})}^{k \in \mathbb{N}} \mathsf{MTL}_k^{\mathcal{I}}$, and $\mathsf{MTL}^{\mathcal{I}} = \bigcup_{\mathcal{I}' \in \mathsf{FCons}(\mathbb{Z})}^{\mathcal{I}' \subseteq \mathcal{I}, k \in \mathbb{N}} \mathsf{MTL}_k^{\mathcal{I}'}$ for each $\mathcal{I} \in \mathsf{Cons}(\mathbb{Z})$.

Lemma 1. *For each $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$ and $k \in \mathbb{N}$, there are only finitely many formulas in $\mathsf{MTL}_k^{\mathcal{I}}$ up to equivalence.*

We define a family of equivalence relations over $(2^{\mathsf{P}} \times \mathbb{N})^{\omega} \times \mathbb{N}$. Let w_0, w_1 be two data words, $i_0, i_1 \ge 0$ be positions in w_0, w_1, respectively. Let $\mathcal{I} \in \mathsf{Cons}(\mathbb{Z})$, and let $k \in \mathbb{N}$. We say that (w_0, i_0) and (w_1, i_1) are $\mathsf{MTL}_k^{\mathcal{I}}$*-equivalent*, written $(w_0, i_0) \equiv_k^{\mathcal{I}} (w_1, i_1)$, if for each formula $\varphi \in \mathsf{MTL}_k^{\mathcal{I}}$, $(w_0, i_0) \models_{\mathsf{MTL}} \varphi$ if and only if $(w_1, i_1) \models_{\mathsf{MTL}} \varphi$.

3 The Ehrenfeucht–Fraïssé Game for MTL

Next we define the Ehrenfeucht–Fraïssé (EF) game for MTL. Let $\mathcal{I} \subseteq \mathsf{FCons}(\mathbb{Z})$, $k \in \mathbb{N}$, w_0, w_1 be two data words and i_0, i_1 be positions in w_0 and w_1, respectively. The k-round MTL EF game on (w_0, i_0) and (w_1, i_1) with respect to $\mathcal{I}$, denoted by $\mathrm{MG}_k^{\mathcal{I}}(w_0, i_0, w_1, i_1)$, is played by two players, called Spoiler and Duplicator, on the pair (w_0, w_1) of data words starting from the positions i_0 in w_0 and i_1 in w_1.

In each round of the game, Spoiler chooses a word and a position, and Duplicator tries to find a position in the respective other word satisfying conditions concerning the propositional variables and the data values in w_0 and w_1. We say that i_0 and i_1 *agree in the propositional variables* if $(w_0, i_0) \models_{\mathsf{MTL}} p$ iff $(w_1, i_1) \models_{\mathsf{MTL}} p$ for each $p \in \mathsf{P}$. We say that $m, n \in \mathbb{Z}$ *are in the same region* with respect to $\mathcal{I}$ if (a, b) or $[a, a]$ is the smallest interval I such that $a, b \in \mathcal{I}$ and $m \in I$, then $n \in I$. For example, let $\mathcal{I} = \{-\infty, 1, 3, 8, +\infty\}$, 1 and 2 are not in the same region with respect to $\mathcal{I}$, 4 and 5 are in the same region with respect to $\mathcal{I}$.

$\mathrm{MG}_k^{\mathcal{I}}(w_0, i_0, w_1, i_1)$ is defined inductively as follows. If $k = 0$, there are no rounds to be played, Spoiler wins if i_0 and i_1 do not agree in the propositional variables. Otherwise, Duplicator wins. If $k > 0$, in the first round,

1. Spoiler wins this round if i_0 and i_1 do not agree in the propositional variables. Otherwise, he chooses a word w_l ($l \in \{0, 1\}$), and a position $i_l' > i_l$ in w_l.
2. Then Duplicator tries to choose a position $i'_{(1-l)} > i_{(1-l)}$ in $w_{(1-l)}$ such that i_0' and i_1' agree in the propositional variables, and $d_{i_0'} - d_{i_0}$ and $d_{i_1'} - d_{i_1}$ are in the same region with respect to $\mathcal{I}$. If one of the conditions is violated, then Spoiler wins the round.

3. Then, Spoiler has two options: either he chooses to start a new game $\mathrm{MG}^{\mathcal{I}}_{k-1}(w_0, i'_0, w_1, i'_1)$; or
4. Spoiler chooses a position $i_{(1-l)} < i''_{(1-l)} < i'_{(1-l)}$ in $w_{(1-l)}$. In this case Duplicator tries to respond by choosing a position $i_l < i''_l < i'_l$ in w_l such that i''_0 and i''_1 agree in the propositional variables. If this condition is violated, Spoiler wins the round.
5. If Spoiler cannot win in Step 1, 2 or 4, then Duplicator wins this round. Then Spoiler chooses to start a new game $\mathrm{MG}^{\mathcal{I}}_{k-1}(w_0, i''_0, w_1, i''_1)$.

We say that Duplicator has a *winning strategy* for the game $\mathrm{MG}^{\mathcal{I}}_k(w_0, i_0, w_1, i_1)$ if she can win every round of the game regardless of the choices of Spoiler. We denote this by $(w_0, i_0) \sim^{\mathcal{I}}_k (w_1, i_1)$. Otherwise we say that Spoiler has a winning strategy. It follows easily that if $(w_0, i_0) \sim^{\mathcal{I}}_k (w_1, i_1)$, then for all $m < k$, $(w_0, i_0) \sim^{\mathcal{I}}_m (w_1, i_1)$.

Theorem 1. *For each* $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$ *and* $k \in \mathbb{N}$, $(w_0, i_0) \equiv^{\mathcal{I}}_k (w_1, i_1)$ *if and only if* $(w_0, i_0) \sim^{\mathcal{I}}_k (w_1, i_1)$.

Theorem 2. *Let* $\mathbf{L}$ *be a data language. The following are equivalent:*

1. $\mathbf{L}$ *is not definable in* MTL.
2. *For each* $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$ *and* $k \in \mathbb{N}$ *there exist* $w_0 \in \mathbf{L}$ *and* $w_1 \notin \mathbf{L}$ *such that* $(w_0, 0) \sim^{\mathcal{I}}_k (w_1, 0)$.

4 Application of the EF Game for MTL

4.1 Relative Expressiveness of TPTL and MTL

In this section, we present one of the main results in this paper: Over data words, TPTL is strictly more expressive than MTL. Before we come to this result, we show in the following lemma that in a data word the difference between data values is what matters, as opposed to the specific numerical value.

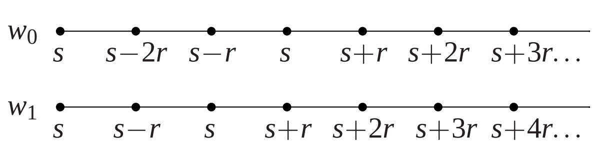
Lemma 2. *Let* $w_0 = (P_0, d_0)(P_1, d_1)\ldots$ *and* $w_1 = (P_0, d_0 + c)(P_1, d_1 + c)\ldots$ *for some* $c \in \mathbb{N}$ *be two data words. Then for every* $k \in \mathbb{N}$ *and* $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$, $(w_0, 0) \sim^{\mathcal{I}}_k (w_1, 0)$.

Proof. The proof is straightforward. If Spoiler chooses a position in w_l ($l \in \{0,1\}$), then the duplicator can respond with the same position in $w_{(1-l)}$. □

From now on, we use $(w_l : i, w_{(1-l)} : j)\,(l \in \{0,1\})$ to denote that Spoiler chooses a word w_l and a position i in w_l and Duplicator responds with a position j in $w_{(1-l)}$.

Proposition 1. *The* $\mathsf{UnaFreezeLTL}^1$*-formula* $x.\mathsf{FFF}(x = 0)$ *and the* TPTL*-formula* $x.\mathsf{F}(b \wedge \mathsf{F}(c \wedge x \leq 2))$ *are not definable in* MTL.

Proof. To show that the formula $\varphi = x.\mathsf{FFF}(x = 0)$ is not definable in MTL, for each $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$ and $k \in \mathbb{N}$, we will define two data words w_0 and w_1 such that $w_0 \models \varphi$ and $w_1 \not\models \varphi$, and $(w_0, 0) \sim^{\mathcal{I}}_k (w_1, 0)$. Then, by Theorem 2, φ is not definable in MTL. So let $r, s \in \mathbb{N}$ be such that all numbers in $\mathcal{I}$ are contained in $(-r, +r)$ and $s \geq 2r$. Intuitively, we choose r in such a way that a jump of magnitude $\pm r$ in data value cannot be detected by $\mathsf{MTL}^{\mathcal{I}}$, as all constants in $\mathcal{I}$ are smaller than r. Define w_0 and w_1 as follows:

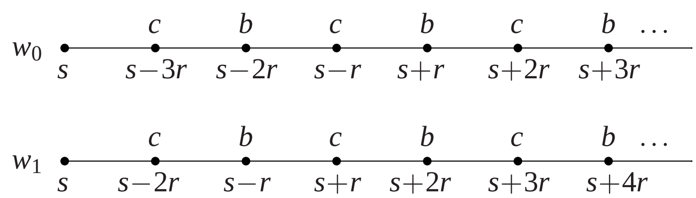


There are no propositional variables in w_0, w_1. We show that Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, 0, w_1, 0)$. The case $k = 0$ is trivial. Suppose $k > 0$. Note that after the first round, they start a new $(k-1)$-round game $\mathrm{MG}_{k-1}^{\mathcal{I}}(w_0, i_0, w_1, i_1)$, where $i_0, i_1 \geq 1$. By Lemma 2, Duplicator has a winning strategy for this game. So it is sufficient to show that Duplicator can win the first round. In the following we give the winning strategy for Duplicator in the first round.

Move \ Case	1	2	3	4
1st	$(w_l : 1, w_{(1-l)} : 1)$, $(l \in \{0,1\})$	$(w_0 : 2, w_1 : 1)$	$(w_0 : i, w_1 : i-1)$, $(i > 2)$	$(w_1 : i, w_0 : i+1)$, $(i \geq 2)$
2nd	-	-	$(w_1 : j, w_0 : j+1)$, $(0 < j < i-1)$	$(w_0 : 1, w_1 : 1)$, or $(w_0 : j, w_1 : j-1)$, $(2 \leq j < i+1)$

By the choice of number r, $d_1^{w_0} - d_0^{w_0} (= -2r)$ is in the same region as $d_1^{w_1} - d_0^{w_1} (= -r)$. It is easy to check that Duplicator's responses satisfy the winning condition about the data value. Hence $(w_0, 0) \sim_k^{\mathcal{I}} (w_1, 0)$.

The proof for the formula $x.\mathsf{F}(b \wedge \mathsf{F}(c \wedge x \leq 2))$ is similar, we define $\mathcal{I}$, k, r and $s \geq 3r$ as above. We leave it to the reader to verify that Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, 0, w_1, 0)$ on the following two data words.



□

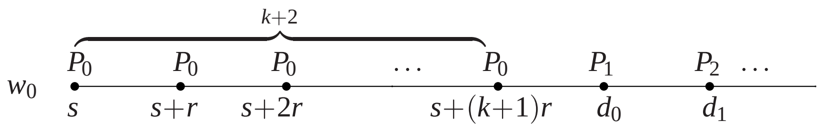
As a corollary, together with the fact that every MTL-formula is equivalent to a TPTL^1-formula we obtain the following.

Corollary 1. TPTL^1 *is strictly more expressive than* MTL.

4.2 The MTL Definability Decision Problem

For many logics whose expressiveness has been shown to be in a strict inclusion relation, the definability decision problem has been considered. For example, it is well known that Monadic second-order logic (MSO) defines exactly regular languages. Its first-order fragment (FO) defines the star-free languages which is a proper subset of regular languages. The problem of whether a MSO formula is equivalent to an FO formula over words is decidable. In our case the problem is stated as follows: Given a TPTL-formula φ, is φ definable in MTL? We show in the following, using the EF game method, that this problem is undecidable. First, we prove a Lemma.

Lemma 3. *Given an arbitrary* $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$*, let* $r, s \in \mathbb{N}$ *be such that all numbers in* $\mathcal{I}$ *are contained in* $(-r, +r)$*. For each* $k \in \mathbb{N}$*, if the data word* w_0 *is of the following form:*



where $P_i \subseteq \mathsf{P}, d_i \geq s+(k+2)r, (i \geq 0)$, *and* w_1 *is defined by* $w_1 = w_0[1]$, *then Duplicator has a winning strategy on the game* $\mathrm{MG}_k^{\mathcal{I}}(w_0, 0, w_1, 0)$.

Proof. The proof is by induction on k. It is trivial when $k = 0$. Suppose the statement holds for k, we must show that it also holds for $k+1$, *i.e.*, Duplicator has a winning strategy for the game $\mathrm{MG}_{k+1}^{\mathcal{I}}(w_0, 0, w_1, 0)$. We give the winning strategy for Duplicator as follows:

- $(w_l : 1, w_{(1-l)} : 1), (l \in \{0, 1\})$. Then, by induction hypothesis, Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, 1, w_1, 1)$.
- $(w_0 : i, w_1 : i-1), (i \geq 2)$. Then by Lemma 2, Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, i, w_1, i-1)$. Moreover, for the second move of Spoiler in this round, if $(w_1 : j, w_0 : j+1)$, $(0 < j < i-1)$, by Lemma 2, Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, j+1, w_1, j)$.
- $(w_1 : i, w_0 : i+1), (i \geq 2)$. Then by Lemma 2, Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, i+1, w_1, i)$. Moreover, for the second move, if $(w_0 : 1, w_1 : 1)$, by induction hypothesis, Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, 1, w_1, 1)$. Otherwise, if $(w_0 : j, w_1 : j-1), (1 < j < i+1)$, by Lemma 2, Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, j, w_1, j-1)$.

This completes the proof.

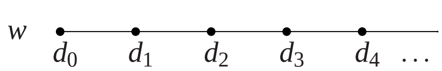
□

Theorem 3. *The problem, whether a given* TPTL*-formula is definable in* MTL*, is undecidable.*

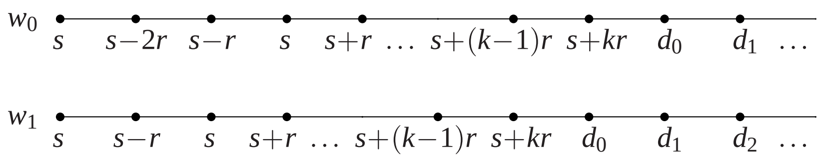
Proof. The recurrent state problem for two-counter machines is defined as follows: given a two-counter machine M, does there exist a computation of M that visits the initial instruction infinitely often? Alur and Henzinger showed that this problem is Σ_1^1-hard [3]. We reduce the recurrent state problem to the MTL definability decision problem in the following way: For each two-counter machine M, we construct a TPTL-formula ψ_M such that ψ_M is definable in MTL iff M is a negative instance of the recurrent state problem.

We use the fact that for each two-counter machine M there is a TPTL-formula φ_M which is satisfiable iff M is a positive instance of the recurrent state problem [3]. Define $\psi_M = (x.\mathsf{FFF}(x = 0)) \wedge \mathsf{F}\varphi_M$. If φ_M is unsatisfiable, then ψ_M is definable by the MTL-formula `False`. Otherwise, if φ_M is satisfiable, we will prove that ψ_M is not definable in MTL. We show that for each $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$ and $k \in \mathbb{N}$, there is no formula in $\mathsf{MTL}_k^{\mathcal{I}}$ that is equivalent to ψ_M.

For an arbitrary $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$, let $r, s \in \mathbb{N}$ be such that all numbers in $\mathcal{I}$ are contained in $(-r, +r)$ and $s \geq 2r$. Suppose $k \geq 1$. By an exploration of the proof in [3] we can find that there is no propositional variable occurring in φ_M, and by Lemma 2, if a data word satisfies φ_M, then the new data word obtained by adding the same arbitrary value to every data value in the original word still satisfies φ_M. Hence we can assume that the data word w satisfying φ_M is of the form:



where $d_i \geq s+(k+1)r$ for each $i \geq 0$. We define the following two data words w_0 and w_1:



Clearly, $w_0 \models_{\mathsf{TPTL}} \psi_M$ and $w_1 \not\models_{\mathsf{TPTL}} \psi_M$. To show that there is no formula in $\mathsf{MTL}_k^{\mathcal{I}}$ that is equivalent to ψ_M, we prove that Duplicator has a winning strategy for the game $\mathrm{MG}_k^{\mathcal{I}}(w_0, 0, w_1, 0)$. The winning strategy for Duplicator in the first round is the same as the one that we give in the proof of Lemma 3. By Lemma 2 and 3, Duplicator can win the remaining rounds.

Since $\mathsf{MTL} = \bigcup_{\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})}^{k \in \mathbb{N}} \mathsf{MTL}_k^{\mathcal{I}}$, we know by the argument given above that there is no formula in MTL that is equivalent to ψ_M if φ_M is satisfiable.

□

4.3 Effects on the Expressiveness of MTL by Restriction of syntactic Resources

We use the EF game for MTL to show the effects of restricting syntactic resources of MTL-formulas. We start with restrictions on the class of constraints occurring in an MTL-formula. For each $n \in \mathbb{Z}$, define $\varphi^n = \mathsf{F}_{[n,n]}\mathtt{True}$.

Lemma 4. *Let $\mathcal{I}_1, \mathcal{I}_2 \in \mathsf{Cons}(\mathbb{Z})$, for each $n \in \mathbb{Z}$, if $n \in \mathcal{I}_1$ and $n-1, n$ or $n, n+1$ are not in $\mathcal{I}_2$, then φ^n is definable in $\mathsf{MTL}^{\mathcal{I}_1}$ but not in $\mathsf{MTL}^{\mathcal{I}_2}$.*

Let $\mathcal{I}[n] = \{m \in \mathbb{Z} \mid m \leq n\} \cup \{-\infty, +\infty\}$. The expressive power relation $\preccurlyeq$ defines a linear order on the set $\{\mathsf{MTL}^{\mathcal{I}[n]} \mid n \in \mathbb{Z}\}$ such that if $n_1 \leq n_2$, then $\mathsf{MTL}^{\mathcal{I}[n_1]} \preccurlyeq \mathsf{MTL}^{\mathcal{I}[n_2]}$. We have $\mathsf{MTL} = \bigcup\{\mathsf{MTL}^{\mathcal{I}[n]} \mid n \in \mathbb{Z}\}$.

Proposition 2. (Linear Constraint Hierarchy of MTL)
For each $n_1, n_2 \in \mathbb{Z}$, if $n_1 < n_2$, then $\mathsf{MTL}^{\mathcal{I}[n_1]} \prec \mathsf{MTL}^{\mathcal{I}[n_2]}$.

In Proposition 2 we show that $\mathsf{MTL}^{\mathcal{I}[n+1]}$ is strictly more expressive than $\mathsf{MTL}^{\mathcal{I}[n]}$. Intuitively, if $\mathcal{I}_2$ is a proper subset of $\mathcal{I}_1$, one should expect that $\mathsf{MTL}^{\mathcal{I}_1}$ is more powerful than $\mathsf{MTL}^{\mathcal{I}_2}$. But in general this is not true. For example, $\mathsf{MTL}^{\mathcal{I}_1}$ with $\mathcal{I}_1 = \{-\infty, 0, 1, 2, +\infty\}$ has the same expressive power as $\mathsf{MTL}^{\mathcal{I}_2}$ where $\mathcal{I}_2 = \mathcal{I}_1 \backslash \{1\}$, since we can use 0 and 2 to express constraints that use the constant 1. It is natural to ask, for $\mathcal{I} \in \mathsf{Cons}(\mathbb{Z})$, what is the minimal subset $\mathcal{I}'$ of $\mathcal{I}$ such that $\mathsf{MTL}^{\mathcal{I}'} \equiv \mathsf{MTL}^{\mathcal{I}}$. In the following we give another constraint hierarchy.

Let EVEN be the subset of $\mathsf{Cons}(\mathbb{Z})$ where only even numbers are in consideration. Let $\mathbf{even} \in \mathsf{EVEN}$ be the set that contains all even numbers. It is easily seen that $\mathsf{MTL}^{\mathbf{even}} \equiv \mathsf{MTL}$. Given $\mathcal{I}_1, \mathcal{I}_2 \in \mathsf{EVEN}$, if $\mathcal{I}_1 \subsetneq \mathcal{I}_2$, by Lemma 4, we have $\mathsf{MTL}^{\mathcal{I}_1} \prec \mathsf{MTL}^{\mathcal{I}_2}$. The expressive power relation $\preccurlyeq$ defines a partial order on the set $\{\mathsf{MTL}^{\mathcal{I}} \mid \mathcal{I} \in \mathsf{EVEN}\}$.

Proposition 3. (Lattice Constraint Hierarchy of MTL)
$\left\langle \{\mathsf{MTL}^{\mathcal{I}} \mid \mathcal{I} \in \mathsf{EVEN}\}, \preccurlyeq \right\rangle$ constitutes a complete lattice in which

(i) the greatest element is $\mathsf{MTL}^{\mathbf{even}}$,

(ii) the least element is $\mathsf{MTL}^{\{-\infty,+\infty\}}$,

and for each nonempty subset $S \subseteq \mathsf{EVEN}$,

(iii) $\bigwedge_{\mathcal{I} \in S} \mathsf{MTL}^{\mathcal{I}} = \mathsf{MTL}^{\bigcap_{\mathcal{I} \in S} \mathcal{I}}$,

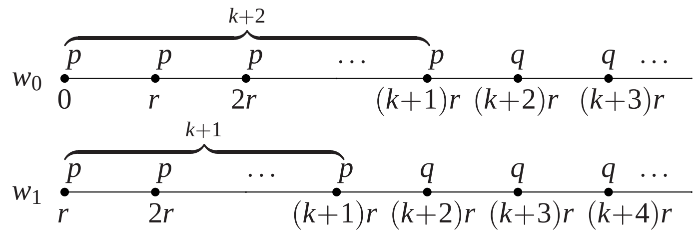
(iv) $\bigvee_{\mathcal{I} \in S} \mathsf{MTL}^{\mathcal{I}} = \mathsf{MTL}^{\bigcup_{\mathcal{I} \in S} \mathcal{I}}$.

Note that $\langle\{\mathsf{MTL}^{\mathcal{I}} \mid \mathcal{I} \in \mathsf{EVEN}\}, \preccurlyeq\rangle$ is isomorphic to the complete lattice $\langle\mathcal{P}(X), \subseteq\rangle$, where X is a countable infinite set, $\mathcal{P}(X)$ is the powerset of X and $\subseteq$ is the containment relation.

Next we show that, as for LTL [14], there is a strict until hierarchy for MTL.

Proposition 4. *For all $k \in \mathbb{N}$, MTL_{k+1} is strictly more expressive than MTL_k.*

Proof. Define $\varphi[1] = (p \wedge \mathsf{X}p)$ and $\varphi[k+1] = (p \wedge \mathsf{X}\varphi[k])$ for every $k \geq 1$. Note that for each $k \geq 1$, $\varphi[k] \in \mathsf{MTL}_k$. We show that for each $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z}), k \geq 0$, $\varphi[k+1]$ is not definable in $\mathsf{MTL}_k^{\mathcal{I}}$. Let $r \in \mathbb{N}$ be such that all numbers in $\mathcal{I}$ are contained in $(-r, +r)$. Define two data words w_0 and w_1 as follows:



We see that $w_0 \models_{\mathsf{MTL}} \varphi[k+1]$ and $w_1 \not\models_{\mathsf{MTL}} \varphi[k+1]$. By Lemma 3 and Theorem 2, there is no formula in $\mathsf{MTL}_k^{\mathcal{I}}$ that is equivalent to $\varphi[k+1]$. Since $\mathsf{MTL}_k = \bigcup_{\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})} \mathsf{MTL}_k^{\mathcal{I}}$, $\varphi[k+1]$ is not definable in MTL_k.

□

As for the MTL definability decision problem, we can show that the MTL_k definability decision problem which asks whether the data language defined by an MTL_{k+1}-formula is definable in MTL_k is undecidable. As a corollary, we know that whether an MTL-formula is equivalent to an MTL_k-formula is undecidable.

Proposition 5. *There exists $m \in \mathbb{N}$ such that for every $k \geq m$, the problem whether a formula $\varphi \in \mathsf{MTL}_{k+1}$ is definable in MTL_k is undecidable.*

5 The Ehrenfeucht-Fraïssé Game for TPTL

In Proposition 1 we have proved that there is an $\mathsf{UnaFreezeLTL}^1$-formula that is not definable in MTL, and we concluded that TPTL^1 is strictly more expressive than MTL. A natural question is to ask for the relation between MTL, UnaTPTL and FreezeLTL. For this, we define the EF game for TPTL.

The *until rank* of a TPTL-formula φ, denoted by $\mathsf{Urk}(\varphi)$, is defined analogously to that of MTL-formulas in Sect. 2.4; we additionally define $\mathsf{Urk}(x \in I) = 0$ and $\mathsf{Urk}(x.\varphi) = \mathsf{Urk}(\varphi)$. Let $\mathcal{I} \in \mathsf{Cons}(\mathbb{Z}), k \geq 0, n \geq 1$, we define

$$\mathsf{TPTL}^{\mathcal{I}} = \{\varphi \in \mathsf{TPTL} \mid \text{ for each subformula } x \in I \text{ of } \varphi, \text{ the endpoints of } I \text{ belong to } \mathcal{I}\},$$
$$\mathsf{TPTL}^{n} = \{\varphi \in \mathsf{TPTL} \mid \text{the register variables in } \varphi \text{ are from } \{x_1, \ldots, x_n\}\},$$
$$\mathsf{TPTL}_k = \{\varphi \in \mathsf{TPTL} \mid \mathsf{Urk}(\varphi) \leq k\}, \quad \mathsf{TPTL}_k^{n,\mathcal{I}} = \mathsf{TPTL}^n \cap \mathsf{TPTL}^{\mathcal{I}} \cap \mathsf{TPTL}_k.$$

Lemma 5. *For each $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$, $n \geq 1$ and $k \geq 0$, there are only finitely many formulas in $\mathsf{TPTL}_k^{n,\mathcal{I}}$ up to equivalence.*

Let w_0, w_1 be two data words, and $i_0, i_1 \geq 0$ be positions in w_0, w_1, respectively, and v_0, v_1 be two register valuations. We say that (w_0, i_0, v_0) and (w_1, i_1, v_1) are $\mathsf{TPTL}_k^{n,\mathcal{I}}$-*equivalent*, written $(w_0, i_0, v_0) \equiv_k^{n,\mathcal{I}} (w_1, i_1, v_1)$, if for each formula $\varphi \in \mathsf{TPTL}_k^{n,\mathcal{I}}$, $(w_0, i_0, v_0) \models_{\mathsf{TPTL}} \varphi$ iff $(w_1, i_1, v_1) \models_{\mathsf{TPTL}} \varphi$.

The k-round TPTL EF game on (w_0, i_0, v_0) and (w_1, i_1, v_1) with respect to n and $\mathcal{I}$, denoted by $\mathrm{TG}_k^{n,\mathcal{I}}(w_0, i_0, v_0, w_1, i_1, v_1)$, is played by Spoiler and Duplicator on w_0 and w_1 starting from i_0 in w_0 with valuation v_0 and i_1 in w_1 with valuation v_1.

We say that (i_0, v_0) and (i_1, v_1) *agree in the atomic formulas in* $\mathsf{TPTL}^{n,\mathcal{I}}$, if $(w_0, i_0, v_0) \models_{\mathsf{TPTL}} p$ iff $(w_1, i_1, v_1) \models_{\mathsf{TPTL}} p$ for for each $p \in \mathsf{P}$, and $(w_0, i_0, v_0) \models_{\mathsf{TPTL}} x \in I$ iff $(w_1, i_1, v_1) \models_{\mathsf{TPTL}} x \in I$ for each formula $x \in I$ in $\mathsf{TPTL}^{n,\mathcal{I}}$.

Analogously to the EF game for MTL, $\mathrm{TG}_k^{n,\mathcal{I}}(w_0, i_0, v_0, w_1, i_1, v_1)$ is defined inductively. If $k = 0$, then Spoiler wins if (i_0, v_0) and (i_1, v_1) do not agree in the atomic formulas in $\mathsf{TPTL}^{n,\mathcal{I}}$. Otherwise, Duplicator wins. Suppose $k > 0$, in the first round,

1. Spoiler wins this round if (i_0, v_0) and (i_1, v_1) do not agree in the atomic formulas in $\mathsf{TPTL}^{n,\mathcal{I}}$. Otherwise, Spoiler chooses a subset Y (maybe empty) of $\{x_1, \ldots, x_n\}$ and sets $v_l' = v_l[x := d_{i_l} (x \in Y)]$ for all $l \in \{0, 1\}$. Then Spoiler chooses a word w_l for some $l \in \{0, 1\}$ and a position $i_l' > i_l$ in w_l.
2. Then Duplicator tries to choose a position $i'_{(1-l)} > i_{(1-l)}$ in $w_{(1-l)}$ such that (i_0', v_0') and (i_1', v_1') agree in the atomic formulas in $\mathsf{TPTL}^{n,\mathcal{I}}$. If Duplicator fails, then Spoiler wins this round.
3. Then, Spoiler has two options: either he chooses to start a new game $\mathrm{TG}_{k-1}^{n,\mathcal{I}}(w_0, i_0', v_0', w_1, i_1', v_1')$; or
4. Spoiler chooses a position $i_{(1-l)} < i''_{(1-l)} < i'_{(1-l)}$ in $w_{(1-l)}$. Then Duplicator tries to respond by choosing a position $i_l < i_l'' < i_l'$ in w_l such that (i_0'', v_0') and (i_1'', v_1') agree in the atomic formulas in $\mathsf{TPTL}^{n,\mathcal{I}}$. If Duplicator fails to do so, Spoiler wins this round.
5. If Spoiler cannot win in Step 1, 2 or 4, then Duplicator wins this round. Then Spoiler chooses to start a new game $\mathrm{TG}_{k-1}^{n,\mathcal{I}}(w_0, i_0'', v_0', w_1, i_1'', v_1')$.

If Duplicator has a winning strategy for the game $\mathrm{TG}_k^{n,\mathcal{I}}(w_0, i_0, v_0, w_1, i_1, v_1)$, then we denote it by $(w_0, i_0, v_0) \sim_k^{n,\mathcal{I}} (w_1, i_1, v_1)$.

Theorem 4. *For each* $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$, $n \geq 1, k \geq 0$, $(w_0, i_0, v_0) \equiv_k^{n,\mathcal{I}} (w_1, i_1, v_1)$ *if and only if* $(w_0, i_0, v_0) \sim_k^{n,\mathcal{I}} (w_1, i_1, v_1)$.

Theorem 5. *Let* $\mathbf{L}$ *be a data language. For each* $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$, $n \geq 1$ *and* $k \geq 0$, *the following are equivalent:*

1. $\mathbf{L}$ *is not definable in* $\mathsf{TPTL}_k^{n,\mathcal{I}}$.
2. *There exist* $w_0 \in \mathbf{L}$ *and* $w_1 \notin \mathbf{L}$ *such that* $(w_0, 0, \overline{0}) \sim_k^{n,\mathcal{I}} (w_1, 0, \overline{0})$.

5.1 More on the Relative Expressiveness of MTL and TPTL

We are going to compare MTL with two fragments of TPTL, namely the unary fragment UnaTPTL and the fragment FreezeLTL. Using the EF game for TPTL we can prove the following results:

Proposition 6. *The* MTL-*formula* $\mathsf{F}_{=1}\mathtt{True}$ *is not definable in* FreezeLTL.

Proposition 7. *The* MTL-*formula* $(\neg a)\mathsf{U}b$ *is not definable in* UnaTPTL.

We remark that for these results, we have to slightly change the definition of the games to suit to the fragments FreezeLTL and UnaTPTL such that an analogous version of Theorem 4 holds. The preceding propositions yield another interesting result for MTL and these two fragments of TPTL.

Corollary 2. *1.* MTL *and* FreezeLTL *are incomparable.*

2. MTL *and* UnaTPTL *are incomparable.*

3. UnaTPTL *and* FreezeLTL *are incomparable.*

Analogously to Theorem 3, we can prove that the FreezeLTL (resp., UnaTPTL) definability problem is undecidable.

Proposition 8. *The problem, whether a given* TPTL*-formula is definable in* FreezeLTL *(resp.,* UnaTPTL*), is undecidable.*

5.2 Restricting Resources in TPTL

In the following we prove results on the effects of restricting syntactic resources of TPTL-formulas similar to those for MTL. For each $n \in \mathbb{Z}$, we redefine $\varphi^n = x.\mathsf{F}(x = n)$.

Lemma 6. *Let* $\mathcal{I}_1, \mathcal{I}_2 \in \mathsf{Cons}(\mathbb{Z})$*, for each* $n \in \mathbb{Z}$*, if* $n \in \mathcal{I}_1$ *and* $n-1, n$ *or* $n, n+1$ *are not in* $\mathcal{I}_2$*, then* φ^n *is definable in* $\mathsf{TPTL}^{\mathcal{I}_1}$ *but not in* $\mathsf{TPTL}^{\mathcal{I}_2}$*.*

Using this lemma we can prove the following two propositions.

Proposition 9. (Linear Constraint Hierarchy of TPTL)
The expressive power relation $\preccurlyeq$ *defines a linear order on the set* $\{\mathsf{TPTL}^{\mathcal{I}[n]} \mid n \in \mathbb{Z}\}$ *such that if* $n_1 \leq n_2$*, then* $\mathsf{TPTL}^{\mathcal{I}[n_1]} \preccurlyeq \mathsf{TPTL}^{\mathcal{I}[n_2]}$ *. Moreover, if* $n_1 < n_2$*, then* $\mathsf{TPTL}^{\mathcal{I}[n_1]} \prec \mathsf{TPTL}^{\mathcal{I}[n_2]}$*.*

Proposition 10. (Lattice Constraint Hierarchy of TPTL)
$\left\langle \{\mathsf{TPTL}^{\mathcal{I}} \mid \mathcal{I} \in \mathsf{EVEN}\}, \preccurlyeq \right\rangle$ *constitutes a complete lattice in which*

(i) *the greatest element is* $\mathsf{TPTL}^{\mathbf{even}} (\equiv \mathsf{TPTL})$*,*

(ii) *the least element is* $\mathsf{TPTL}^{\{-\infty,+\infty\}} (\equiv \mathsf{LTL})$*,*

and for each nonempty subset $S \subseteq \mathsf{EVEN}$*,*

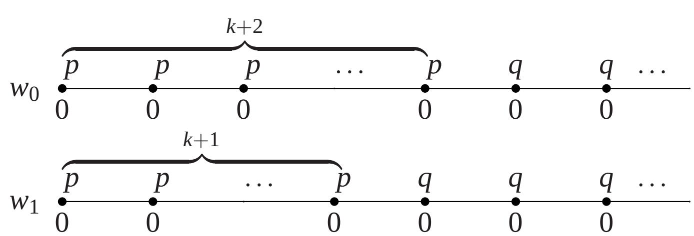
(iii) $\bigwedge_{\mathcal{I} \in S} \mathsf{TPTL}^{\mathcal{I}} = \mathsf{TPTL}^{\bigcap_{\mathcal{I} \in S} \mathcal{I}}$*,*

(iv) $\bigvee_{\mathcal{I} \in S} \mathsf{TPTL}^{\mathcal{I}} = \mathsf{TPTL}^{\bigcup_{\mathcal{I} \in S} \mathcal{I}}$*.*

In the next proposition we show that the until hierarchy for TPTL is strict.

Proposition 11. TPTL_{k+1} *is strictly more expressive than* TPTL_k*.*

Proof. Let $\varphi[k]\,(k \geq 1)$ be as defined in Proposition 4. $\varphi[k]$ is a formula in TPTL_k. For every $k \geq 0$. We can show that $(w_0, 0, \overline{0}) \sim_k^{n,\mathcal{I}} (w_1, 0, \overline{0})$ on the following two data words $w_0 \models_{\mathsf{TPTL}} \varphi[k+1]$ and $w_1 \not\models_{\mathsf{TPTL}} \varphi[k+1]$.



□

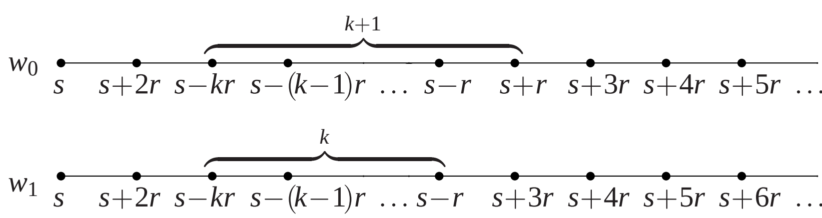
Corollary 3. *MTL_{k+1} and TPTL_k are incomparable in expressive power.*

Proposition 12. *There exists $m \in \mathbb{N}$ such that for every $k \geq m$, the problem whether a formula $\varphi \in \mathsf{TPTL}_{k+1}$ is definable in TPTL_k is undecidable.*

We have seen in the previous chapters that TPTL is strictly more expressive than MTL. The register variables play a crucial role in reaching this greater expressiveness. In the following we want to explore more deeply whether the number of register variables allowed in a TPTL formula has an impact on the expressive power of the logic. We are able to show that there is a strict increase in expressiveness when allowing two register variables instead of just one. The following results concern the number of register variables allowed in a TPTL-formula.

Proposition 13. *For the $\mathsf{UnaTPTL}^2$-formula $\varphi = x_1.\mathsf{F}(x_1 > 0 \wedge x_2.\mathsf{F}(x_1 > 0 \wedge x_2 < 0))$ there is no equivalent formula in TPTL^1.*

Proof. Let $\mathcal{I} \in \mathsf{FCons}(\mathbb{Z})$ and $k \geq 1$. Let $s, r \in \mathbb{N}$ be such that all elements in $\mathcal{I}$ are contained in $(-r, +r)$ and $s - kr \geq 0$. One can show that $(w_0, 0, \overline{0}) \sim_k^{1,\mathcal{I}} (w_1, 0, \overline{0})$ on the following two data words $w_0 \models_{\mathsf{TPTL}} \varphi$ and $w_1 \not\models_{\mathsf{TPTL}} \varphi$.



□

Corollary 4. *TPTL^2 is strictly more expressive than TPTL^1.*

It remains open whether we can generalize this result to TPTL^{n+1} and TPTL^n, where $n \geq 2$, to get a complete hierarchy for the number of register variables. We have the following conjecture.

Conjecture 1. *For each $n \geq 1$, TPTL^{n+1} is strictly more expressive than TPTL^n.*

6 Conclusion and Future Work

In this paper, we consider the expressive power of MTL and TPTL on non-monotonic ω-data words and introduce EF games for these two logics. We show that TPTL is strictly more expressive than MTL and some other expressiveness results of various syntactic restrictions. For TPTL, we examine the effects of allowing only a bounded number of register variables: We prove that TPTL^2 is strictly more expressive than TPTL^1, but it is still open if TPTL^{n+1} is strictly more expressive than TPTL^n for all $n \geq 1$ (Conjecture 1). In future work we want to figure out whether there is a decidable characterization for the set of data domains for which TPTL and MTL are equally expressive.

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