On the Non-Monotonic Description Logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$

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Abstract

In the last 20 years many proposals have been made to incorporate non-monotonic reasoning into description logics, ranging from approaches based on default logic and circumscription to those based on preferential semantics. In particular, the non-monotonic description logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ uses a combination of the preferential semantics with minimization of a certain kind of concepts, which represent atypical instances of a class of elements. One of its drawbacks is that it suffers from the problem known as the *property blocking inheritance*, which can be seen as a weakness from an inferential point of view. In this paper we propose an extension of $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$, namely $\mathcal{ALC}$+$\mathbf{T}^{+}_{\mathsf{min}}$, with the purpose to solve the mentioned problem. In addition, we show the close connection that exists between $\mathcal{ALC}$+$\mathbf{T}^{+}_{\mathsf{min}}$ and concept-circumscribed knowledge bases. Finally, we study the complexity of deciding the classical reasoning tasks in $\mathcal{ALC}$+$\mathbf{T}^{+}_{\mathsf{min}}$.

Introduction.

Description Logics (DLs) (Baader et al. 2003) are a well-investigated family of logic-based knowledge representation formalisms. They can be used to represent knowledge of a problem domain in a structured and formal way. To describe this kind of knowledge each DL provides constructors that allow to build concept descriptions. A knowledge base consists of a TBox that states general assertions about the problem domain and an ABox that asserts properties about explicit individuals.

Nevertheless, classical description logics do not provide any means to reason about exceptions. In the past 20 years research has been directed with the purpose to incorporate non-monotonic reasoning formalisms into DLs. In (Baader and Hollunder 1995a), an integration of Reiter's default logic (Reiter 1980) within the terminological language $\mathcal{ALCF}$ is proposed and later extended in (Baader and Hollunder 1995b) to allow the use of priorities between default rules. Taking a different approach, (Bonatti, Lutz, and Wolter 2009) introduces circumscribed DLs and analyses in detail the complexity of reasoning in circumscribed extensions of expressive description logics. In addition, recent works (Casini and Straccia 2010; Britz, Meyer, and Varzinczak 2011; Giordano et al. 2013a) attempt to introduce defeasible reasoning by extending DLs with preferential and rational semantics based on the KLM approach to propositional non-monotonic reasoning (Lehmann and Magidor 1992).

In particular, the logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ introduced in (Giordano et al. 2013b) combines the use of a preferential semantics and the minimization of a certain kind of concepts. This logic is built on top of the description logic $\mathcal{ALC}$ (Schmidt-Schauß and Smolka 1991) and is based on a typicality operator $\mathbf{T}$ whose intended meaning is to single out the *typical* instances of a class C of elements. The underlying semantics of $\mathbf{T}$ is based on a preference relation over the domain. More precisley, classical $\mathcal{ALC}$ interpretations are equipped with a partial order over the domain elements setting a preference relation among them. Based on such an order, for instance, the set of *typical birds* or $\mathbf{T}(\mathsf{Bird})$, comprises those individuals from the domain that are birds and minimal in the class of all birds with respect to the preference order. Using this operator, the subsumption statement $\mathbf{T}(\mathsf{Bird}) \sqsubseteq \mathsf{Fly}$ expresses that *typical birds fly*. In addition, the use of a minimal model semantics considers models that minimize the atypical instances of Bird. Then, when no information is given about whether a bird is able to fly or not, it is possible to *assume* that it flies in view of the assertion $\mathbf{T}(\mathsf{Bird}) \sqsubseteq \mathsf{Fly}$.

As already pointed out by the authors, the preferential order over the domain limits the logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ in the sense that if a class P is an exceptional case of a superclass B, then no default properties from B can be inherited by P during the reasoning process, including those that are unrelated with the exceptionality of P with respect to B. For example:

$$\begin{aligned} \mathsf{Penguin} &\sqsubseteq \mathsf{Bird} \\ \mathbf{T}(\mathsf{Bird}) &\sqsubseteq \mathsf{Fly} \sqcap \mathsf{Winged} \\ \mathbf{T}(\mathsf{Penguin}) &\sqsubseteq \neg\mathsf{Fly} \end{aligned}$$

It is not possible to infer that typical penguins have wings, even when the only reason for them to be exceptional with respect to birds is that they normally do not fly.

In the present paper we extend the non-monotonic logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ from (Giordano et al. 2013b) with the introduction of several preference relations. We show how this extension can handle the inheritance of defeasible properties,

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resembling the use of abnormality predicates from circumscription (McCarthy 1986). In addition, we show the close relationship between the extended non-monotonic logic $\mathcal{ALC}$+$\mathbf{T}^{+}_{\mathsf{min}}$ and *concept-circumscribed* knowledge bases (Bonatti, Lutz, and Wolter 2009). Based on such a relation, we provide a complexity analysis of the different reasoning tasks showing $\mathsf{NExp}^{\mathsf{NP}}$- completeness for concept satisfiability and co-$\mathsf{NExp}^{\mathsf{NP}}$-completeness for subsumption and instance checking.

Missing proofs can be found in the long version of the paper at http://www.informatik.uni-leipzig.de/~fernandez/NMR14long.pdf.

The logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$.

We recall the logic $\mathcal{ALC}$+$\mathbf{T}$ proposed in (Giordano et al. 2013b) and its non-monotonic extension $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$. Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be three countable sets of *concept names, role names and individual names*, respectively. The language defined by $\mathcal{ALC}$+$\mathbf{T}$ distinguishes between normal concept descriptions and *extended concept* descriptions which are formed according to the following syntax rules:

$$C ::= A \mid \neg C \mid C \sqcap D \mid \exists r.C,$$
$$C_e ::= C \mid \mathbf{T}(A) \mid \neg C_e \mid C_e \sqcap D_e$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, C and D are classical $\mathcal{ALC}$ concept descriptions, C_e and D_e are extended concept descriptions, and $\mathbf{T}$ is the newly introduced operator. We use the usual abbreviations $C \sqcup D$ for $\neg(\neg C \sqcap \neg D)$, $\forall r.C$ for $\neg\exists r.\neg C$, $\top$ for $A \sqcup \neg A$ and $\bot$ for $\neg\top$.

A knowledge base is a pair $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. The TBox $\mathcal{T}$ contains subsumption statements $C \sqsubseteq D$ where C is a classical $\mathcal{ALC}$ concept or an extended concept of the form $\mathbf{T}(A)$, and D is a classical $\mathcal{ALC}$ concept. The Abox $\mathcal{A}$ contains assertions of the form $C_e(a)$ and $r(a, b)$ where C_e is an extended concept, $r \in \mathsf{N_R}$ and $a, b \in \mathsf{N_I}$. The assumption that the operator $\mathbf{T}$ is applied to concept names is without loss of generality. For a complex $\mathcal{ALC}$ concept C, one can always introduce a fresh concept name A_C which can be made equivalent to C by adding the subsumption statements $A_C \sqsubseteq C$ and $C \sqsubseteq A_C$ to the background TBox. Then, $\mathbf{T}(C)$ can be equivalently expressed as $\mathbf{T}(A_C)$.

In order to provide a semantics for the operator $\mathbf{T}$, usual $\mathcal{ALC}$ interpretations are equipped with a preference relation $<$ over the domain elements:

Definition 1 (Interpretation in $\mathcal{ALC}$+$\mathbf{T}$)**.** An $\mathcal{ALC}$+$\mathbf{T}$ interpretation $\mathcal{I}$ is a tuple $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, <)$ where:

- $\Delta^{\mathcal{I}}$ is the domain,
- $\cdot^{\mathcal{I}}$ is an interpretation function that maps concept names to subsets of $\Delta^{\mathcal{I}}$ and role names to binary relations over $\Delta^{\mathcal{I}}$,
- $<$ is an irreflexive and transitive relation over $\Delta^{\mathcal{I}}$ that satisfies the following condition (**Smoothness Condition**): for all $S \subseteq \Delta^{\mathcal{I}}$ and for all $x \in S$, either $x \in \mathsf{Min}_<(S)$ or $\exists y \in \mathsf{Min}_<(S)$ such that $y < x$, with $\mathsf{Min}_<(S) = \{x \in S \mid \nexists y \in S \text{ s.t. } y < x\}$.

The operator $\mathbf{T}$ is interpreted as follows: $[\mathbf{T}(A)]^{\mathcal{I}} = \mathsf{Min}_<(A^{\mathcal{I}})$. For arbitrary concept descriptions, $\cdot^{\mathcal{I}}$ is inductively extended in the same way as for $\mathcal{ALC}$ taking into account the introduced semantics for $\mathbf{T}$.

As mentioned in (Giordano et al. 2013b; 2009), $\mathcal{ALC}$+$\mathbf{T}$ is still monotonic and has several limitations. In the following we present the logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$, proposed in (Giordano et al. 2013b) as a non-monotonic extension of $\mathcal{ALC}$+$\mathbf{T}$, where a preference relation is defined between $\mathcal{ALC}$+$\mathbf{T}$ interpretations and only minimal models are considered.

First, we introduce the modality $\Box$ as in (Giordano et al. 2013b).

Definition 2. Let $\mathcal{I}$ be an $\mathcal{ALC}$+$\mathbf{T}$ interpretation and C a concept description. Then, $\Box C$ is interpreted under $\mathcal{I}$ in the following way:

$$(\Box C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \text{ for all } y \in \Delta^{\mathcal{I}} \text{ if } y < x \text{ then } y \in C^{\mathcal{I}}\}$$

We remark that $\Box C$ does not extend the syntax of $\mathcal{ALC}$+$\mathbf{T}$. The purpose of using it is to characterize elements of the domain with respect to whether all their predecessors in $<$ are instances of C or not. For example, $\Box\neg\mathsf{Bird}$ defines a concept such that $d \in (\Box\neg\mathsf{Bird})^{\mathcal{I}}$ if all the predecessors of d, with respect to $<$ under the interpretation $\mathcal{I}$, are not instances of Bird. Hence, it is not difficult to see that:

$$[\mathbf{T}(\mathsf{Bird})]^{\mathcal{I}} = (\mathsf{Bird} \sqcap \Box\neg\mathsf{Bird})^{\mathcal{I}}$$

Then, the idea is to prefer models that minimize the instances of $\neg\Box\neg\mathsf{Bird}$ in order to minimize the number of *atypical* birds.

Now, let $\mathcal{L}_{\mathbf{T}}$ be a finite set of concept names occurring in the knowledge base. These are the concepts whose atypical instances are meant to be minimized. For each interpretation $\mathcal{I}$, the set $\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}}}$ represents all the instance of concepts of the form $\neg\Box\neg A$ for all $A \in \mathcal{L}_{\mathbf{T}}$. Formally,

$$\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}}} = \{(x, \neg\Box\neg A) \mid x \in (\neg\Box\neg A)^{\mathcal{I}}, \text{ with } x \in \Delta^{\mathcal{I}}, A \in \mathcal{L}_{\mathbf{T}}\}$$

Based on this, the notion of minimal models is defined in the following way.

Definition 3 (Minimal models)**.** Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, <_{\mathcal{I}})$, $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}}, <_{\mathcal{J}})$ be two interpretations. We say that $\mathcal{I}$ is preferred to $\mathcal{J}$ with respect to the set $\mathcal{L}_{\mathbf{T}}$ (denoted as $\mathcal{I} <_{\mathcal{L}_{\mathbf{T}}} \mathcal{J}$), iff:

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$,
- $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all $a \in \mathsf{N_I}$,
- $\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}}} \subset \mathcal{J}^{\Box^-}_{\mathcal{L}_{\mathbf{T}}}$.

An interpretation $\mathcal{I}$ is a minimal model of $\mathcal{K}$ with respect to $\mathcal{L}_{\mathbf{T}}$ (denoted as $\mathcal{I} \models^{\mathcal{L}_{\mathbf{T}}}_{\mathsf{min}} \mathcal{K}$) iff $\mathcal{I} \models \mathcal{K}$ and there is no model $\mathcal{J}$ of $\mathcal{K}$ such that $\mathcal{J} <_{\mathcal{L}_{\mathbf{T}}} \mathcal{I}$.

Based on the notion of minimal models, the standard reasoning tasks are defined for $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$.

- *Knowledge base consistency (or satisfiability):* A knowledge base $\mathcal{K}$ is consistent w.r.t. $\mathcal{L}_{\mathbf{T}}$, if there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models^{\mathcal{L}_{\mathbf{T}}}_{\mathsf{min}} \mathcal{K}$.

- *Concept satisfiability:* An extended concept C_e is satisfiable with respect to $\mathcal{K}$ if there exists a minimal model $\mathcal{I}$ of $\mathcal{K}$ w.r.t. $\mathcal{L}_{\mathbf{T}}$ such that $C_e^{\mathcal{I}} \neq \emptyset$.
- *Subsumption:* Let C_e and D_e be two extended concepts. C_e is subsumed by D_e w.r.t. $\mathcal{K}$ and $\mathcal{L}_{\mathbf{T}}$, denoted as $\mathcal{K} \models_{\mathsf{min}}^{\mathcal{L}_{\mathbf{T}}} C_e \sqsubseteq D_e$, if $C_e^{\mathcal{I}} \subseteq D_e^{\mathcal{I}}$ for all minimal models $\mathcal{I}$ of $\mathcal{K}$.
- *Instance checking:* An individual name a is an instance of an extended concept C_e w.r.t. $\mathcal{K}$, denoted as $\mathcal{K} \models_{\mathsf{min}}^{\mathcal{L}_{\mathbf{T}}} C_e(a)$, if $a^{\mathcal{I}} \in C_e^{\mathcal{I}}$ in all the minimal models $\mathcal{I}$ of $\mathcal{K}$.

Regarding the computational complexity, the case of *knowledge base consistency* is not interesting in itself since the logic $\mathcal{ALC}$+**T** enjoys the finite model property (Giordano et al. 2013b). Note that if there exists a finite model $\mathcal{I}$ of $\mathcal{K}$, then the sets that are being minimized are finite. Therefore, every descending chain starting from $\mathcal{I}$ with respect to $<_{\mathcal{L}_{\mathbf{T}}}$ must be finite and a minimal model of $\mathcal{K}$ always exists. Thus, the decision problem only requires to decide knowledge base consistency of the underlying monotonic logic $\mathcal{ALC}$+**T** which has been shown to be EXPTIME-complete (Giordano et al. 2009). For the other reasoning tasks, a $\mathsf{NExp}^{\mathsf{NP}}$ upper bound is provided for *concept satisfiability* and a co-$\mathsf{NExp}^{\mathsf{NP}}$ upper bound for subsumption and instance checking (Giordano et al. 2013b).

Extending $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ with more typicality operators.

As already mentioned in (Giordano et al. 2013b; 2009), the use of a global relation to represent that one individual is more typical than another one, limits the expressive power of the logic. It is not possible to express that an individual x is more typical than an individual y with respect to some aspect As_1 and at the same time y is more typical than x (or not comparable to x) with respect to a different aspect As_2. This, for example, implies that a subclass cannot inherit any property from a superclass, if the subclass is already exceptional with respect to one property of the superclass. This effect is also known as *property inheritance blocking* (Pearl 1990; Geffner and Pearl 1992), and is a known problem in preferential extensions of DLs based on the KLM approach.

We revisit the example from the introduction to illustrate this problem.

Example 4. Consider the following knowledge base:

$$\mathsf{Penguin} \sqsubseteq \mathsf{Bird}$$
$$\mathbf{T}(\mathsf{Bird}) \sqsubseteq \mathsf{Fly} \sqcap \mathsf{Winged}$$
$$\mathbf{T}(\mathsf{Penguin}) \sqsubseteq \neg\mathsf{Fly}$$

Here, *penguins* represent an exceptional subclass of *birds* in the sense that they *usually are unable to fly*. However, it might be intuitive to conclude that they *normally have wings* ($\mathbf{T}(\mathsf{Penguin}) \sqsubseteq \mathsf{Winged}$) since although birds fly because they have wings, having wings does not imply the ability to fly. In fact, as said before, it is not possible to sanction this kind of conclusion in $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$. The problem is that due to the global character of the order $<$ among individuals of the domain, once an element d is assumed to be a *typical* penguin, then automatically a more preferred individual e must exist that is a *typical* bird. This rules out the possibility to apply the non-monotonic assumption represented by the second assertion to d.

In relation with circumscription, this situation can be modelled using abnormality predicates to represent exceptionality with respect to different aspects (McCarthy 1980; 1986). The following example shows a knowledge base which is defined using abnormality concepts similar as the examples in (Bonatti, Lutz, and Wolter 2009).

Example 5.

$$\mathsf{Penguin} \sqsubseteq \mathsf{Bird}$$
$$\mathsf{Bird} \sqsubseteq \mathsf{Fly} \sqcup Ab_1$$
$$\mathsf{Bird} \sqsubseteq \mathsf{Winged} \sqcup Ab_2$$
$$\mathsf{Penguin} \sqsubseteq \neg\mathsf{Fly} \sqcup Ab_{\mathsf{penguin}}$$

The semantics of circumscription allows to consider only models that minimize the instances of the abnormality concepts. In this example, concepts Ab_1 and Ab_2 are used to represent birds that are atypical with respect to two independent aspects (i.e.: Fly and Winged). If the minimization forces an individual d to be a *not abnormal* penguin (i.e.: d is not an instance of Ab_{penguin}), then it must be an instance of Ab_1, but at the same time nothing forces it to be an instance of Ab_2. Therefore, it is possible to assume that d has wings because of the minimization of Ab_2.

In this paper, we follow a suggestion given in (Giordano et al. 2013b) that asks for the extension of the logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ with more preferential relations in order to express typicality of a class with respect to different aspects. We define the logic $\mathcal{ALC}$+$\mathbf{T}^+$ and its extension $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ in a similar way as for $\mathcal{ALC}$+**T** and $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$, but taking into account the possibility to use more than one typicality operator.

We start by fixing a finite number of typicality operators $\mathbf{T}_1, \ldots, \mathbf{T}_k$. Classical concept descriptions and extended concept descriptions are defined by the following syntax:

$$C ::= A \mid \neg C \mid C \sqcap D \mid \exists r.C,$$
$$C_e ::= C \mid \mathbf{T}_i(A) \mid \neg C_e \mid C_e \sqcap D_e,$$

where all the symbols have the same meaning as in $\mathcal{ALC}$+**T** and $\mathbf{T}_i$ ranges over the set of typicality operators. The semantics is defined as an extension of the semantics for $\mathcal{ALC}$+**T** that takes into account the use of more than one **T** operator.

Definition 6 (Interpretations in $\mathcal{ALC}$+$\mathbf{T}^+$)**.** An interpretation $\mathcal{I}$ in $\mathcal{ALC}$+$\mathbf{T}^+$ is a tuple $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, <_1, \ldots, <_k)$ where:

- $\Delta^{\mathcal{I}}$ is the domain,
- $<_i$ $(1 \leq i \leq k)$ is an irreflexive and transitive relation over $\Delta^{\mathcal{I}}$ satisfying the Smoothness Condition.

Typicality operators are interpreted in the expected way with respect to the different preference relations over the domain: $[\mathbf{T}_i(A)]^{\mathcal{I}} = \mathsf{Min}_{<_i}(A^{\mathcal{I}})$.

Similar as for $\mathcal{ALC}$+**T**, we introduce for each preference relation $<_i$ an indexed box modality $\Box_i$ such that:

$$(\Box_i C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}} : \text{ if } y <_i x \text{ then } y \in C^{\mathcal{I}}\}$$

Then, the set of typical instances of a concept A with respect to the i^{th} typical operator can be expressed in terms of the indexed $\Box$ modalities:

$$[\mathbf{T}_i(A)]^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid x \in (A \sqcap \Box_i \neg A)^{\mathcal{I}}\}$$

Now, we define the extension of $\mathcal{ALC}$+$\mathbf{T}^+$ that results in the non-monotonic logic $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$. Let $\mathcal{L}_{\mathbf{T}_1}, \dots, \mathcal{L}_{\mathbf{T}_k}$ be k finite sets of concept names. Given an $\mathcal{ALC}$+$\mathbf{T}^+$ interpretation $\mathcal{I}$, the sets $\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}_i}}$ are defined as:

$$\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}_i}} = \{(x, \neg\Box_i\neg A) \mid x \in (\neg\Box_i\neg A)^{\mathcal{I}} \wedge A \in \mathcal{L}_{\mathbf{T}_i}\}$$

Based on these sets, we define the preference relation $<^+_{\mathcal{L}_\mathbf{T}}$ on $\mathcal{ALC}$+$\mathbf{T}^+$ interpretations that characterizes the non-monotonic semantics of $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$.

Definition 7 (Preference relation)**.** Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}}, <_{i_1}, \dots, <_{i_k})$, $\mathcal{J} = (\Delta^{\mathcal{J}}, .^{\mathcal{J}}, <_{j_1}, \dots, <_{j_k})$ be two interpretations. We say that $\mathcal{I}$ is preferred to $\mathcal{J}$ (denoted as $<^+_{\mathcal{L}_\mathbf{T}}$) with respect to the sets $\mathcal{L}_{\mathbf{T}_i}$, iff:

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$,
- $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all $a \in \mathsf{N_I}$,
- $\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}_i}} \subseteq \mathcal{J}^{\Box^-}_{\mathcal{L}_{\mathbf{T}_i}}$ for all $1 \le i \le k$,
- $\exists \ell$ s.t. $\mathcal{I}^{\Box^-}_{\mathcal{L}_{\mathbf{T}_\ell}} \subset \mathcal{J}^{\Box^-}_{\mathcal{L}_{\mathbf{T}_\ell}}$.

An $\mathcal{ALC}$+$\mathbf{T}^+$ interpretation $\mathcal{I}$ is a minimal model of $\mathcal{K}$ (denoted as $\mathcal{I} \models^{\mathcal{L}_{\mathbf{T}^+}}_{\mathsf{min}} \mathcal{K}$) iff $\mathcal{I} \models \mathcal{K}$ and there exists no interpretation $\mathcal{J}$ such that: $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J} <^+_{\mathcal{L}_\mathbf{T}} \mathcal{I}$. The different reasoning tasks are defined in the usual way, but with respect to the new entailment relation $\models^{\mathcal{L}_{\mathbf{T}^+}}_{\mathsf{min}}$.

We revise Example 4 to show how to distinguish between a bird being typical with respect *to being able to fly* or *to having wings*, in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$. The example shows the use of two typicality operators $\mathbf{T}_1$ and $\mathbf{T}_2$, where $<_1$ and $<_2$ are the underlying preference relations.

Example 8.

$$\mathsf{Penguin} \sqsubseteq \mathsf{Bird}$$
$$\mathbf{T}_1(\mathsf{Bird}) \sqsubseteq \mathsf{Fly}$$
$$\mathbf{T}_2(\mathsf{Bird}) \sqsubseteq \mathsf{Winged}$$
$$\mathbf{T}_1(\mathsf{Penguin}) \sqsubseteq \neg\mathsf{Fly}$$

In the example, we use two preference relations to express typicality of birds with respect to two different aspects independently. The use of a second preference relation permits that typical penguins can also be typical birds with respect to $<_2$. Therefore, it is possible to infer that typical penguins do have wings. Looking from the side of individual elements: having the assertion $\mathsf{Penguin}(e)$, the minimal model semantics allows to assume that e is a typical penguin and also a typical bird with respect to $<_2$, even when a bird d must exist such that d is preferred to e with respect to $<_1$.

It is interesting to observe that the defeasible property *not being able to fly*, for penguins, is stated with respect to $\mathbf{T}_1$. If instead, we use $\mathbf{T}_2(\mathsf{Penguin}) \sqsubseteq \neg\mathsf{Fly}$, there will be minimal models where e is an instance of $\mathbf{T}_1(\mathsf{Bird})$ and others where it is an instance of $\mathbf{T}_2(\mathsf{Penguin})$. This implies that it will not be possible to infer for e, the defeasible properties corresponding to the most specific concept it belongs to.

The same problem is realized, with respect to circumscription in Example 5, where some minimal models prefer e to be a *normal* bird ($e \in \neg Ab_1$), while others consider e as a *normal* penguin ($e \in \neg Ab_{\mathsf{penguin}}$). To address this problematic about specificity, one needs to use priorities between the minimized concepts (or abnormality predicates) (McCarthy 1986; Bonatti, Lutz, and Wolter 2009).

In contrast, for the formulation in the example, the semantics induced by the preferential order $<_1$ does not allow to have interpretations where $e \in \mathsf{Penguin}$, $e \in \mathbf{T}_1(\mathsf{Bird})$ and $e \notin \mathbf{T}_1(\mathsf{Penguin})$, i.e., the treatment of specificity comes for free in the semantics of the logic.

Complexity of reasoning in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$.

In the following, we show that reasoning in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ is $\mathsf{NExp}^{\mathsf{NP}}$-complete for *concept satisfiability* and co-$\mathsf{NExp}^{\mathsf{NP}}$-complete for *subsumption* and *instance checking*. As a main tool we use the close correspondence that exists between *concept-circumscribed* knowledge bases in the DL $\mathcal{ALC}$ (Bonatti, Lutz, and Wolter 2009) and $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ knowledge bases. In fact, this relation has been pointed out in (Giordano et al. 2013b) with respect to the logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$. However, on the one hand, the provided mapping from $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ into *concept-circumscribed* knowledge bases is not polynomial, and instead a tableaux calculus is used to show the upper bounds for the main reasoning tasks in $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$. On the other hand, the relation in the opposite direction is only given with respect to the logic $\mathcal{ALCO}$+$\mathbf{T}_{\mathsf{min}}$, which extends $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ by allowing the use of nominals.

First, we improve the mapping proposed in (Giordano et al. 2013b) by giving a simpler polynomial reduction, that translates $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ knowledge bases into *concept-circumscribed* knowledge bases while preserving the entailment relation under the translation. Second, we show that using more than one typicality operator, it is possible to reduce the problem of concept satisfiability for *concept-circumscribed* knowledge bases in $\mathcal{ALC}$, into the concept satisfiability problem for $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$.

We start by introducing circumscribed knowledge bases in the DL $\mathcal{ALC}$, as defined in (Bonatti, Lutz, and Wolter 2009). We obviate the use of priorities between minimized predicates.

Definition 9. A circumscribed knowledge base is an expression of the form $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$ where $\mathsf{CP} = (M, F, V)$ is a circumscription pattern such that M, F, V partition the predicates (i.e.: concept and role names) used in $\mathcal{T}$ and $\mathcal{A}$. The set M identifies those concept names whose extension is minimized, F those whose extension must remain fixed and V those that are free to vary. A circumscribed knowledge base where $M \cup F \subseteq \mathsf{N_C}$ is called a *concept-circumscribed* knowledge base.

To formalize a semantics for circumscribed knowledge bases, a preference relation $<_{\mathsf{CP}}$ is defined on interpretations by setting $\mathcal{I} <_{\mathsf{CP}} \mathcal{J}$ iff:

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$,

- $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all $a \in \mathsf{N_I}$,
- $A^{\mathcal{I}} = A^{\mathcal{J}}$ for all $A \in F$,
- $A^{\mathcal{I}} \subseteq A^{\mathcal{J}}$ for all $A \in M$ and there exists an $A' \in M$ such that $A'^{\mathcal{I}} \subset A'^{\mathcal{J}}$.

An interpretation $\mathcal{I}$ is a model of $\mathsf{Circ_{CP}}(\mathcal{T}, \mathcal{A})$ if $\mathcal{I}$ is a model of $(\mathcal{T}, \mathcal{A})$ and there is no model $\mathcal{I}'$ of $(\mathcal{T}, \mathcal{A})$ with $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$. The different reasoning tasks can be defined in the same way as above.

Similar as for circumscribed knowledge bases in (Bonatti, Lutz, and Wolter 2009), one can show that concept satisfiability, subsumption and instance checking can be polynomially reduced to one another in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$. However, to reduce instance checking into concept satisfiability slightly different technical details have to be considered.

Lemma 10. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$+$\mathbf{T}^+$ knowledge base, C_e an extended concept, $\mathcal{L}_{\mathbf{T}_1}, \ldots, \mathcal{L}_{\mathbf{T}_k}$ be finite sets of concept names and A a fresh concept name not occurring in $\mathcal{K}$ and C_e. Then, $\mathcal{K} \models^{\mathcal{L}_{\mathcal{T}^+}}_{\mathsf{min}} C_e(a)$ iff $\neg\mathbf{T}_{k+1}(A) \sqcap \neg C_e$ is unsatisfiable w.r.t. $\mathcal{K}' = (\mathcal{T} \cup \{\top \sqsubseteq A\}, \mathcal{A} \cup \{(\neg\mathbf{T}_{k+1}(A))(a)\})$, where $\mathcal{L}_{\mathbf{T}_{k+1}} = \{A\}$.*

Note that this reduction requires the introduction of an additional typicality operator $\mathbf{T}_{k+1}$. Nevertheless, this does not represent a problem in terms of complexity since, as it will be shown in the following, the complexity does not depend on the number of typicality operators k whenever $k \geq 2$.

Upper Bound.

Before going into the details of the reduction we need to define the notion of a signature.

Definition 11. Let $\mathsf{N_T}$ be the set of all the concepts of the form $\mathbf{T}_i(A)$ where $A \in \mathsf{N_C}$. A signature Σ for $\mathcal{ALC}$+$\mathbf{T}^+$ is a finite subset of $\mathsf{N_C} \cup \mathsf{N_R} \cup \mathsf{N_T}$. We denote by $\Sigma|_{\mathcal{ALC}}$ the set $\Sigma \setminus \mathsf{N_T}$.

The signature $\mathsf{sig}(C_e)$ of an extended concept C_e is the set of all concept names, role names and concepts from $\mathsf{N_T}$ that occur in C_e. Similarly, the signature $\mathsf{sig}(\mathcal{K})$ of an $\mathcal{ALC}$+$\mathbf{T}^+$ knowledge base $\mathcal{K}$ is the union of the signatures of all concept descriptions occurring in $\mathcal{K}$. Finally, we denote by $\mathsf{sig}(E_1, \ldots, E_m)$ the set $\mathsf{sig}(E_1) \cup \ldots \cup \mathsf{sig}(E_m)$, where each E_i is either an extended concept or a knowledge base.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$+$\mathbf{T}^+$ knowledge base, $\mathcal{L}_{\mathbf{T}_1}, \ldots, \mathcal{L}_{\mathbf{T}_k}$ finite sets of concept names and Σ be any signature with $\mathsf{sig}(\mathcal{K}) \subseteq \Sigma$. A corresponding circumscribed knowledge base $\mathsf{Circ_{CP}}(\mathcal{T}', \mathcal{A}')$, with $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$, is built in the following way:

- For every concept A such that it belongs to some set $\mathcal{L}_{\mathbf{T}_i}$ or $\mathbf{T}_i(A) \in \Sigma$, a fresh concept name A^*_i is introduced. These concepts are meant to represent the atypical elements with respect to A and $<_i$ in $\mathcal{K}$, i.e., $\neg\Box_i\neg A$.
- Every concept description C defined over Σ is transformed into a concept $\bar{C}$ by replacing every occurrence of $\mathbf{T}_i(A)$ by $(A \sqcap \neg A^*_i)$.
- The TBox $\mathcal{T}'$ is built as follows:
  - $\bar{C} \sqsubseteq \bar{D} \in \mathcal{T}'$ for all $C \sqsubseteq D \in \mathcal{T}$,
  - For each new concept A^*_i the following assertions are included in $\mathcal{T}'$:
$$A^*_i \equiv \exists r_i.(A \sqcap \neg A^*_i) \quad (1)$$
$$\exists r_i.A^*_i \sqsubseteq A^*_i \quad (2)$$
    where r_i is a fresh role symbol, not occurring in Σ, introduced to represent the relation $<_i$.
- $\mathcal{A}'$ results from replacing every assertion of the form $C(a)$ in $\mathcal{T}$ by the assertion $\bar{C}(a)$.
- Let $\mathcal{L}_{\mathbf{T}}$ be the set:
$$\bigcup_{j=1}^{k} \bigcup_{A \in \mathcal{L}_{\mathbf{T}_j}} A^*_j$$
then, the concept circumscription pattern CP is defined as $\mathsf{CP} = (M, F, V) = (\mathcal{L}_{\mathbf{T}}, \emptyset, \Sigma|_{\mathcal{ALC}} \cup \{A^*_i \mid A^*_i \notin \mathcal{L}_{\mathbf{T}}\} \cup \{r_i \mid 1 \leq i \leq k\})$.

One can easily see that the provided encoding is polynomial in the size of $\mathcal{K}$. The use of the signature Σ is just a technical detail and since it is chosen arbitrarily, one can also select it properly for the encoding of the different reasoning tasks.

The idea of the translation is to simulate each order $<_i$ with a relation r_i and at the same time fulfill the semantics underlying the $\mathbf{T}_i$ operators. The first assertion, $A^*_i \equiv \exists r_i.(A \sqcap \neg A^*_i)$, intends to express that the atypical elements with respect to A and $<_i$ are those, and only those, that have an r_i-successor e that is an instance of A and at the same time a not atypical A, i.e. , $e \in \mathbf{T}_i(A)$. Indeed, this is a consequence from the logic $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ because the order $<_i$ is transitive. However, since it is not possible to enforce transitivity of r_i when translated into $\mathcal{ALC}$, we need to use the second assertion $\exists r_i.A^*_i \sqsubseteq A^*_i$. This prevents to have the following situation:

$$d \in A^*_1 \quad d \in B \sqcap \neg B^*_1 \quad (d, e) \in r_1 \quad e \in A \sqcap \neg A^*_1 \quad e \in B^*_1$$

In the absence of assertion (2), this would be consistent with respect to $\mathcal{T}'$, but it would not satisfy the aim of the translation since the typical B-element d would have a predecessor (r_i-successor) e which is atypical with respect to B. In fact, the translation provided in (Giordano et al. 2013b) also deals with this situation, but all the possible cases are asserted explicitly yielding an exponential encoding.

The following auxiliary lemma shows that a model of $(\mathcal{T}', \mathcal{A}')$ can always be transformed into a model, that only differs in the interpretation of r_i, and $(r_i)^{-1}$ is irreflexive, transitive and *well-founded*.

Lemma 12. *Let $\mathcal{I}$ be an $\mathcal{ALC}$ interpretation such that $\mathcal{I} \models (\mathcal{T}', \mathcal{A}')$. Then, there exists $\mathcal{J}$ such that $\mathcal{J} \models (\mathcal{T}', \mathcal{A}')$, $X^{\mathcal{I}} = X^{\mathcal{J}}$ for all $X \in \Sigma|_{\mathcal{ALC}} \cup \bigcup A^*_i$, and for each r_i we have: $\left(r_i{}^{\mathcal{J}}\right)^{-1}$ is irreflexive, transitive and well-founded.*

Since *well-foundedness* implies the Smoothness Condition, the previous lemma allows us to assume (without loss of generality) that $\left(r_i{}^{\mathcal{I}}\right)^{-1}$ is irreflexive, transitive and satisfies the Smoothness Condition for every model $\mathcal{I}$ of $\mathcal{K}'$.

Now, we denote by $\mathcal{M}_\mathcal{K}$ the set of models of $\mathcal{K}$ and by $\mathcal{M}_{\mathcal{K}'}$ the set of models of $\mathcal{K}'$. With the help of the previous lemma, we show that there exists a *one-to-one* correspondence between $\mathcal{M}_\mathcal{K}$ and $\mathcal{M}_{\mathcal{K}'}$. We start by defining a mapping φ that transforms $\mathcal{ALC}$+$\mathbf{T}^+$ interpretations into $\mathcal{ALC}$ interpretations.

Definition 13. We define a mapping φ from $\mathcal{ALC}$+$\mathbf{T}^+$ interpretations into $\mathcal{ALC}$ interpretations such that $\varphi(\mathcal{I}) = \mathcal{J}$ iff:

- $\Delta^\mathcal{J} = \Delta^\mathcal{I}$,
- $X^\mathcal{J} = X^\mathcal{I}$ for each $X \in \Sigma|_{\mathcal{ALC}}$,
- $(A_i^*)^\mathcal{J} = (\neg\Box_i\neg A)^\mathcal{I}$ for each fresh concept name A_i^*,
- $(r_i)^\mathcal{J} = (<_i)^{-1}$ for all $i, 1 \leq i \leq k$,
- $a^\mathcal{J} = a^\mathcal{I}$, for all $a \in \mathsf{N_I}$.

Remark. We stress that interpretations are considered only with respect to concept and role names occurring in Σ for $\mathcal{ALC}$+$\mathbf{T}^+$, and $\Sigma|_{\mathcal{ALC}} \cup \{A_i^*\} \cup \{r_i\}$ for $\mathcal{ALC}$. All the other concept and role names from $\mathsf{N_C}$ and $\mathsf{N_R}$ are not relevant to distinguish one interpretation from another one. This is, if $\mathcal{I}$ and $\mathcal{J}$ are two $\mathcal{ALC}$+$\mathbf{T}^+$ interpretations, then $\mathcal{I} \equiv \mathcal{J}$ iff $X^\mathcal{I} = X^\mathcal{J}$ for all $X \in \Sigma \cap (\mathsf{N_C} \cup \mathsf{N_R})$ and $(<_i)^\mathcal{I} = (<_i)^\mathcal{J}$ for all $i, 1 \leq i \leq k$. The same applies for $\mathcal{ALC}$ interpretations, but with respect to $\Sigma|_{\mathcal{ALC}} \cup \{A_i^*\} \cup \{r_i\}$.

Next, we show that φ is indeed a bijection from $\mathcal{M}_\mathcal{K}$ to $\mathcal{M}_{\mathcal{K}'}$.

Lemma 14. *The mapping φ is a bijection from $\mathcal{M}_\mathcal{K}$ to $\mathcal{M}_{\mathcal{K}'}$, such that for every $\mathcal{I} \in \mathcal{M}_\mathcal{K}$ and each extended concepts C_e defined over Σ: $C_e^\mathcal{I} = (\bar{C}_e)^{\varphi(\mathcal{I})}$.*

Proof. First, we show that for each $\mathcal{I} \in \mathcal{M}_\mathcal{K}$ it holds that: $\varphi(\mathcal{I}) \in \mathcal{M}_{\mathcal{K}'}$. Let $\mathcal{I} = (\Delta^\mathcal{I}, .^\mathcal{I}, <_1, \ldots, <_k)$ be a model of $\mathcal{K}$ and assume that $\varphi(\mathcal{I}) = \mathcal{J}$. We observe that since $[\mathbf{T}_i(A)]^\mathcal{I} = (A \sqcap \Box_i\neg A)^\mathcal{I}$, then by definition of φ it follows that:

$$[\mathbf{T}_i(A)]^\mathcal{I} = (A \sqcap \neg A_i^*)^\mathcal{J} \tag{3}$$

Consequently, one can also see that for every extended concept C_e defined over Σ and every element $d \in \Delta^\mathcal{I}$:

$$d \in C_e^\mathcal{I} \text{ iff } d \in (\bar{C}_e)^\mathcal{J} \tag{4}$$

This can be shown by a straightforward induction on the structure of C_e where the base cases are A and $\mathbf{T}_i(A)$. Hence, it follows that $C_e^\mathcal{I} = (\bar{C}_e)^\mathcal{J}$ for every extended concept C_e defined over Σ.

Now, we show that $\mathcal{J} \models (\mathcal{T}', \mathcal{A}')$. From (4), it is clear that $\mathcal{J} \models \bar{C} \sqsubseteq \bar{D}$ for all $\bar{C} \sqsubseteq \bar{D} \in \mathcal{T}'$. In addition, since $a^\mathcal{J} = a^\mathcal{I}$ for all $a \in \mathsf{N_I}$, $\mathcal{J}$ satisfies each assertion in $\mathcal{A}'$. It is left to show that each GCI in $\mathcal{T}'$ containing an occurrence of a fresh role r_i is also satisfied by $\mathcal{J}$. For each $d \in \Delta^\mathcal{I}$ and concept name A_i^*, it holds:

$$\begin{aligned} d \in (A_i^*)^\mathcal{J} &\text{iff } d \in (\neg\Box_i\neg A)^\mathcal{I} \\ &\text{iff } \exists e \in \Delta^\mathcal{I} \text{ s.t. } e <_i d \text{ and } e \in [\mathbf{T}_i(A)]^\mathcal{I} \\ &\text{iff } (d,e) \in (r_i)^\mathcal{J} \text{ and } e \in (A \sqcap \neg A_i^*)^\mathcal{J} \quad \text{by (3)} \\ &\text{iff } d \in (\exists r_i.(A \sqcap \neg A_i^*))^\mathcal{J} \end{aligned}$$

The case for the second GCI ($\exists r_i.A_i^* \sqsubseteq A_i^*$) can be shown in a very similar way. Thus, $\mathcal{J} \models (\mathcal{T}', \mathcal{A}')$ and consequently φ is a function from $\mathcal{M}_\mathcal{K}$ into $\mathcal{M}_{\mathcal{K}'}$.

Second, we show that for any model $\mathcal{J}$ of $\mathcal{K}'$ (i.e. $\mathcal{J} \in \mathcal{M}_{\mathcal{K}'}$), there exists $\mathcal{I} \in \mathcal{M}_\mathcal{K}$ with $\varphi(\mathcal{I}) = \mathcal{J}$. Let $\mathcal{J}$ be an arbitrary model of $\mathcal{K}'$, we build an $\mathcal{ALC}$+$\mathbf{T}^+$ interpretation $\mathcal{I} = (\Delta^\mathcal{I}, .^\mathcal{I}, <_1, \ldots, <_k)$ in the following way:

- $\Delta^\mathcal{I} = \Delta^\mathcal{J}$,
- $X^\mathcal{I} = X^\mathcal{J}$ for each $X \in \Sigma|_{\mathcal{ALC}}$,
- $<_i = \left(r_i{}^\mathcal{J}\right)^{-1}$ for all $i, 1 \leq i \leq k$,
- $a^\mathcal{I} = a^\mathcal{J}$, for all $a \in \mathsf{N_I}$.

Next, we show that $(\neg\Box_i\neg A)^\mathcal{I} = (A_i^*)^\mathcal{J}$. Assume that $d \in (\neg\Box_i\neg A)^\mathcal{I}$ for some $d \in \Delta^\mathcal{I}$, then there exists $e <_i d$ such that $e \in A^\mathcal{I}$ and $e \in [\mathbf{T}_i(A)]^\mathcal{I}$. This means that for all $f <_i e$(or $(e,f) \in r_i^\mathcal{J}$): $f \notin A^\mathcal{I}$. Hence, $e \in A^\mathcal{J}$ and $e \notin (A_i^*)^\mathcal{J}$. All in all, we have $(d,e) \in r_i^\mathcal{J}$ and $e \in (A \sqcap \neg A_i^*)^\mathcal{J}$, therefore $d \in (A_i^*)^\mathcal{J}$. Conversely, assume that $d \in (A_i^*)^\mathcal{J}$. Assertion (1) in $\mathcal{T}'$ implies that there exists e such that $(d,e) \in (r_i)^\mathcal{J}$ and $e \in A^\mathcal{J}$. By construction of $\mathcal{I}$ we have $e <_i d$ and $e \in A^\mathcal{I}$. Thus, $d \in (\neg\Box_i\neg A)^\mathcal{I}$ and we can conclude that $(\neg\Box_i\neg A)^\mathcal{I} = (A_i^*)^\mathcal{J}$. Having this, it follows that $\varphi(\mathcal{I}) = \mathcal{J}$. In addition, similar as for equation (3), we have:

$$[\mathbf{T}_i(A)]^\mathcal{I} = (A \sqcap \neg A_i^*)^\mathcal{J} \tag{5}$$

A similar reasoning, as above yields that $\mathcal{I} \models \mathcal{K}$. This implies that φ is surjective. It is not difficult to see, from the definition of φ, that it is also injective. Thus, φ is a bijection from $\mathcal{M}_\mathcal{K}$ to $\mathcal{M}_{\mathcal{K}'}$. □

The previous lemma establishes a *one to one* correspondence between $\mathcal{M}_\mathcal{K}$ and $\mathcal{M}_{\mathcal{K}'}$. Then, since $\mathcal{K}$ is an arbitrary $\mathcal{ALC}$+$\mathbf{T}^+$ knowledge base, Lemma 14 also implies that knowledge base consistency in $\mathcal{ALC}$+$\mathbf{T}^+$ can be polynomially reduced to knowledge base consistency in $\mathcal{ALC}$, which is EXPTIME-complete (Baader et al. 2003).

Theorem 15. *In $\mathcal{ALC}$+$\mathbf{T}^+$, deciding knowledge base consistency is* EXPTIME*-complete.*

In addition, since $\mathcal{ALC}$ enjoys the finite model property, this is also the case for $\mathcal{ALC}$+$\mathbf{T}^+$. Using the same argument given before for $\mathcal{ALC}$+$\mathbf{T}$ and $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$, deciding knowledge base consistency in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ reduces to the same problem with respect to the underlying monotonic logic $\mathcal{ALC}$+$\mathbf{T}^+$. Therefore, we obtain the following theorem.

Theorem 16. *In $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$, deciding knowledge base consistency is* EXPTIME*-complete.*

Now, we show that φ is not only a bijection from $\mathcal{M}_\mathcal{K}$ to $\mathcal{M}_{\mathcal{K}'}$, but it is also *order-preserving* with respect to $<^+_{\mathcal{L}_\mathbf{T}}$ and $<_{\mathsf{CP}}$.

Lemma 17. *Let $\mathcal{I}$ and $\mathcal{J}$ be two models of $\mathcal{K}$. Then, $\mathcal{I} <^+_{\mathcal{L}_\mathbf{T}} \mathcal{J}$ iff $\varphi(\mathcal{I}) <_{\mathsf{CP}} \varphi(\mathcal{J})$.*

Proof. Assume that $\mathcal{I} <^+_{\mathcal{L}_\mathbf{T}} \mathcal{J}$. Then, for all $A \in \mathcal{L}_{T_i}$ we have that $(\neg\Box_i\neg A)^\mathcal{I} \subseteq (\neg\Box_i\neg A)^\mathcal{J}$ and in particular, for some j and $A' \in \mathcal{L}_{T_j}$ we have $(\neg\Box_j\neg A')^\mathcal{I} \subset (\neg\Box_j\neg A')^\mathcal{J}$.

By definition of φ, we know that $(\neg\Box_i\neg A)^\mathcal{I} = (A_i^*)^{\varphi(\mathcal{I})}$. Hence, for all $A_i^* \in M$ we have that $(A_i^*)^{\varphi(\mathcal{I})} \subseteq (A_i^*)^{\varphi(\mathcal{J})}$ and $(A'^*_j)^{\varphi(\mathcal{I})} \subset (A'^*_j)^{\varphi(\mathcal{J})}$. Thus, $\varphi(\mathcal{I}) <_{\mathsf{CP}} \varphi(\mathcal{J})$. The other direction can be shown in the same way. □

The following lemma is an easy consequence from the previous one and the fact that φ is bijection (which implies that φ is invertible).

Lemma 18. *Let $\mathcal{I}$ and $\mathcal{J}$ be $\mathcal{ALC}$+**T**$^+$ and $\mathcal{ALC}$ interpretations, respectively. Then,*

$$\mathcal{I} \models^{\mathcal{L}_{T^+}}_{\mathsf{min}} \mathcal{K} \;\; \textit{iff} \;\; \varphi(\mathcal{I}) \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}') \quad \text{(a)}$$

$$\mathcal{J} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}') \;\; \textit{iff} \;\; \varphi^{-1}(\mathcal{J}) \models^{\mathcal{L}_{T^+}}_{\mathsf{min}} \mathcal{K} \quad \text{(b)}$$

Thus, we have a correspondence between minimal models of $\mathcal{K}$ and models of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}')$. Based on this, it is easy to reduce each reasoning task from $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ into the equivalent task with respect to *concept-circumscribed* knowledge bases. The following lemma states the existence of such a reduction for concept satisfiability, the cases for subsumption and instance checking can be proved in a very similar way.

Lemma 19. *An extended concept C_0 is satisfiable w.r.t. to $\mathcal{K}$ and $\mathcal{L}_{\mathbf{T}_1}, \ldots, \mathcal{L}_{\mathbf{T}_k}$ iff $\bar{C}_0$ is satisfiable in* $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}')$.

Proof. Let us define Σ as $\mathsf{sig}(\mathcal{K}, C_0)$.

($\Rightarrow$) Assume that $\mathcal{I}$ is a minimal model of $\mathcal{K}$ with $C_0^\mathcal{I} \neq \emptyset$. The application of Lemma 18 tells us that $\varphi(\mathcal{I}) \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}')$. In addition, from Lemma 14 we have that $C_0^\mathcal{I} = (\bar{C}_0)^{\varphi(\mathcal{I})}$. Thus, $\bar{C}_0$ is satisfiable in $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}')$. ($\Leftarrow$) The argument is similar, but using φ^{-1}. □

Finally, from the complexity results proved in (Bonatti, Lutz, and Wolter 2009) for the different reasoning tasks with respect to *concept-circumscribed* knowledge bases in $\mathcal{ALC}$, we obtain the following upper bounds.

Theorem 20. *In $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$, it is in* $\mathsf{NExp}^{\mathsf{NP}}$ *to decide concept satisfiability and in co-*$\mathsf{NExp}^{\mathsf{NP}}$ *to decide subsumption and instance checking.*

Lower Bound.

To show the lower bound, we reduce the problem of concept satisfiability with respect to *concept-circumscribed* knowledge bases in $\mathcal{ALC}$, into the concept satisfiability problem in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$. It is enough to consider *concept-circumscribed* knowledge bases of the form $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$ with $\mathsf{CP} = (M, F, V)$ where $\mathcal{A} = \emptyset$ and $F = \emptyset$. The problem of deciding concept satisfiability for this class of circumscribed knowledge bases has been shown to be $\mathsf{NExp}^{\mathsf{NP}}$-hard for $\mathcal{ALC}$ (Bonatti, Lutz, and Wolter 2009). In order to do that, we modify the reduction provided in (Giordano et al. 2013b) which shows $\mathsf{NExp}^{\mathsf{NP}}$-hardness for concept satisfiability in $\mathcal{ALCO}$+$\mathbf{T}_{\mathsf{min}}$.

Before going into the details, we assume without loss of generality that each minimized concept occurs in the knowledge base:

Remark. Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$ be a circumscribed knowledge base. If $A \in M$ and A does not occur in $(\mathcal{T}, \mathcal{A})$, then for each model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$: $A^\mathcal{I} = \emptyset$.

Given a circumscribed knowledge base $\mathcal{K} = \mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$ (where CP is of the previous form) and a concept description C_0, we define a corresponding $\mathcal{ALC}$+$\mathbf{T}^+$ knowledge base $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$ using **two** typicality operators in the following way.

Let M be the set $\{M_1, \ldots, M_q\}$. Similarly as in (Giordano et al. 2013b), individual names c and c_{m_i} (one for each $M_i \in M$) and a fresh concept name D are introduced. Each $\mathcal{ALC}$ concept description C is transformed into C^* inductively by introducing D into concept descriptions of the form $\exists r.C_1$, i.e.: $(\exists r.C_1)^* = \exists r.(D \sqcap C_1^*)$ (see (Giordano et al. 2013b) for precise details).

Similar as in (Giordano et al. 2013b), we start by adding the following GCIs to the TBox $\mathcal{T}'$:

$$D \sqcap C_1^* \sqsubseteq C_2^* \text{ if } C_1 \sqsubseteq C_2 \in \mathcal{T} \quad (6)$$

$$D \sqcap M_i \sqsubseteq \neg\mathbf{T}_1(M_i) \text{ for all } M_i \in M \quad (7)$$

The purpose of using these subsumption statements is to establish a correspondence between the minimized concept names M_i, from the circumscription side, with the underlying concepts $\neg\Box_1\neg M_i$ on the $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ side, such that the minimization of the M_i concepts can be simulated by the minimization of $\neg\Box_1\neg M_i$. The individual names c_{m_i} are introduced to guarantee the existence of typical M_i's in view of assertion (7). The concept D plays the role to distinguish the elements of the domain that are not mapped to those individual names by an interpretation.

Note that if under an interpretation $\mathcal{I}$ an element d is an instance of D and M_i at the same time, then it has to be an instance of $\neg\mathbf{T}_1(M_i)$ and therefore an instance of $\neg\Box_1\neg M_i$ as well. Hence, it is important that whenever d becomes an instance of $\Box_1\neg M_i$ in a preferred interpretation to $\mathcal{I}$, it happens because d becomes an instance of $\neg M_i$ while it is still an instance of D. In order to force this effect during the minimization, the interpretation of the concept D should remain fixed in some way. As pointed out in (Giordano et al. 2013b), this seems not to be possible in $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ and that is why the reduction is realized for $\mathcal{ALCO}$+$\mathbf{T}_{\mathsf{min}}$ where nominals are used with that purpose.

In contrast, for $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ this effect on D can be simulated by introducing a second typicality operator $\mathbf{T}_2$, setting $\mathcal{L}_{\mathbf{T}_1} = M, \mathcal{L}_{\mathbf{T}_2} = \{A\}$ and adding the following two assertions to $\mathcal{T}'$:

$$\top \sqsubseteq A \quad (8)$$

$$\neg D \sqsubseteq \neg\mathbf{T}_2(A) \quad (9)$$

where A is a fresh concept name. Note that if an element d becomes a $(\neg D)$-element, it automatically becomes a $(\neg\Box_2\neg A)$-element.

The ABox $\mathcal{A}'$ contains the following assertions:

- $D(c)$,
- for each $M_i \in M$:
  - $(\neg D)(c_{m_i})$,
  - $(\mathbf{T}_1(M_i))(c_{m_i})$,
  - $(\neg M_j)(c_{m_i})$ for all $j \neq i$.

Finally, a concept description C_0' is defined as $D \sqcap C_0^*$.

Lemma 21. *C_0 is satisfiable in $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$ iff C_0' is satisfiable w.r.t. $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$ in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$.*

Proof. Details of the proof are deferred to the long version of the paper. □

Since the size of $\mathcal{K}'$ is polynomial with respect to the size of $\mathcal{K}$, the application of the previous lemma yields the following result.

Theorem 22. *In $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$, concept satisfiability is $\mathsf{NExp}^{\mathsf{NP}}$-hard.*

Since concept satisfiability, subsumption and instance checking are polynomially interreducible (see Lemma 10), Theorem 22 yields co-$\mathsf{NExp}^{\mathsf{NP}}$ lower bounds for the subsumption and the instance checking problem.

Corollary 23. *In $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$, it is $\mathsf{NExp}^{\mathsf{NP}}$-complete to decide concept satisfiability and co-$\mathsf{NExp}^{\mathsf{NP}}$-complete to decide subsumption and instance checking.*

Finally, we remark that the translations provided between $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ and *concept-circumscribed* knowledge bases do not depend on the classical constructors of the description logic $\mathcal{ALC}$. Therefore, the same translations can be used for the more expressive description logics $\mathcal{ALCIO}$ and $\mathcal{ALCQO}$. From the complexity results obtained in (Bonatti, Lutz, and Wolter 2009) for circumscription in $\mathcal{ALCIO}$ and $\mathcal{ALCQO}$, we also obtain the following corollary.

Corollary 24. *In $\mathcal{ALCIO}$+$\mathbf{T}^+_{\mathsf{min}}$ and $\mathcal{ALCQO}$+$\mathbf{T}^+_{\mathsf{min}}$, it is $\mathsf{NExp}^{\mathsf{NP}}$-complete to decide concept satisfiability and co-$\mathsf{NExp}^{\mathsf{NP}}$-complete to decide subsumption and instance checking.*

Moreover, from the lower bound obtained in (Giordano et al. 2013b) for $\mathcal{ALCO}$+$\mathbf{T}_{\mathsf{min}}$, the results also apply for the logics $\mathcal{ALCIO}$+$\mathbf{T}_{\mathsf{min}}$ and $\mathcal{ALCQO}$+$\mathbf{T}_{\mathsf{min}}$.

Corollary 25. *In $\mathcal{ALCIO}$+$\mathbf{T}_{\mathsf{min}}$ and $\mathcal{ALCQO}$+$\mathbf{T}_{\mathsf{min}}$, it is $\mathsf{NExp}^{\mathsf{NP}}$-complete to decide concept satisfiability and co-$\mathsf{NExp}^{\mathsf{NP}}$-complete to decide subsumption and instance checking.*

Conclusions

In this paper, we have provided an extension of the non-monotonic description logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$, by adding the possibility to use more than one preference relation over the domain elements. This extension, called $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$, allows to express typicality of a class of elements with respect to different aspects in an "independent" way. Based on this, a class of elements P that is exceptional with respect to a superclass B regarding a specific aspect, could still be not exceptional with respect to different unrelated aspects. The latter permits that defeasible properties from B not conflicting with the exceptionality of P, can be inherited by elements in P. As already observed in the paper, this is not possible in the logic $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$.

In addition, we have introduced translations that show the close relationship between $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ and *concept-circumscribed* knowledge bases in $\mathcal{ALC}$. First, the provided translation from $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ into *concept-circumscribed* knowledge bases is polynomial, in contrast with the exponential translation given in (Giordano et al. 2013b) for $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$. Second, the translation presented for the opposite direction shows how to encode circumscribed knowledge base, by using two typicality operators and no nominals.

Using these translations, we were able to determine the complexity of deciding the different reasoning tasks in $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$. We have shown that it is $\mathsf{NExp}^{\mathsf{NP}}$-complete to decide concept satisfiability and co-$\mathsf{NExp}^{\mathsf{NP}}$-complete to decide subsumption and instance checking. Moreover, the same translations can be used for the corresponding extensions of $\mathcal{ALC}$+$\mathbf{T}^+_{\mathsf{min}}$ into more expressive description logics like $\mathcal{ALCIO}$ and $\mathcal{ALCQO}$. The results also apply for extensions of $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ with respect to the underlying description logics, in view of the hardness result shown for $\mathcal{ALCO}$+$\mathbf{T}_{\mathsf{min}}$ in (Giordano et al. 2013b).

As possible future work, the exact complexity for reasoning in $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{min}}$ still remains open. It would be interesting to see if it is actually possible to improve the $\mathsf{NExp}^{\mathsf{NP}}$(co-$\mathsf{NExp}^{\mathsf{NP}}$) upper bounds. If that were the case, there is a possibility to identify a corresponding fragment from *concept-circumscribed* knowledge bases with a better complexity than $\mathsf{NExp}^{\mathsf{NP}}$(co-$\mathsf{NExp}^{\mathsf{NP}}$).

As a different aspect, it can be seen that the logic $\mathcal{ALC}$+$\mathbf{T}$ and our proposed extension $\mathcal{ALC}$+$\mathbf{T}^+$ impose syntactic restrictions on the use of the typicality operator. First, it is not possible to use a typicality operator under a role operator. Second, only subsumption statements of the form $\mathbf{T}(A) \sqsubseteq C$ are allowed in the TBox. The latter, seems to come from the fact that $\mathcal{ALC}$+$\mathbf{T}$ is based on the approach to propositional non-monotonic reasoning proposed in (Lehmann and Magidor 1992), where a conditional assertion of the form $A {\mid\!\sim} C$ is used to express that A's *normally* have property C.

As an example, by lifting these syntactic restrictions, one will be able to express things like:

$$\mathbf{T}(\mathsf{Senior\_Teacher}) \sqsubseteq \mathsf{Excellent\_Teacher}$$

$$\mathbf{T}(\mathsf{Student}) \sqsubseteq \forall \mathsf{attend}.(\mathsf{Class} \sqcap \exists \mathsf{imparted}.\mathbf{T}(\mathsf{Senior\_Teacher}))$$

This allows to relate the typical instances from different classes in a way which is not possible with the current syntax. From a complexity point of view, it is not difficult to observe that the given translations in the paper will also be applicable in this case, without increasing the overall complexity. The reason is that after lifting the mentioned syntactic restrictions, the occurrences of $\mathbf{T}_i(A)$ in an extended concept can still be seen as basic concepts.

Therefore, it would be interesting to study what are the effects of removing these restrictions, with respect to the kind of conclusions that would be obtained from a knowledge base expressed in the resulting non-monotonic logic.

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