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concept

Institute of Theoretical Computer Science Chair of Automata Theory

THE INCLUSION PROBLEM FOR WEIGHTED AUTOMATA ON INFINITE TREES

Stefan Borgwardt Rafael Peñaloza

Debrecen, August 21, 2011

Introduction

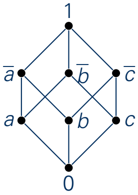
- Automata on infinite trees can recognize tree-shaped models
- Emptiness test useful to decide satisfiability in logics
- Inclusion test could be used to decide entailment
- Here: generalization to lattice-weighted automata

Lattices

De Morgan lattice:

- Bounded distributive lattice $L = (L, \oplus, \otimes, 0, 1)$
- De Morgan negation $^{-} : L \to L$

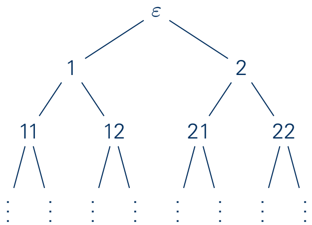
$\overline{x} = 1 - x$



Trees

Infinite k-ary trees:

- Nodes are identified by their positions in K^*, where $K := \{1, \ldots, k\}$



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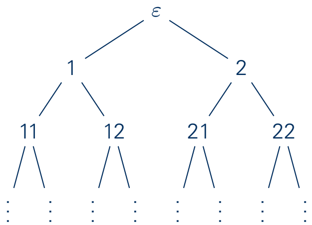

- A labeled tree $t \in \Sigma^{K^*}$ is a function $t : K^* \to \Sigma$

Automata

Tree automaton $\mathcal{A} = (Q, \Sigma, I, \Delta, \mathfrak{X})$:

- states Q
- input alphabet Σ
- initial state set $I \subseteq Q$
- transition relation $\Delta \subseteq Q \times \Sigma \times Q^k$
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Acceptance conditions: looping, Büchi, co-Büchi, parity

Automata (2)

PTIME-problems for Büchi automata:

- Infimum of two automata: $(\|\mathcal{C}\|, t) = (\|\mathcal{A}\|, t) \otimes (\|\mathcal{B}\|, t)$
- Supremum of two automata: $(\|\mathcal{C}\|, t) = (\|\mathcal{A}\|, t) \oplus (\|\mathcal{B}\|, t)$
- Computing the behavior $\bigoplus_{t \in \Sigma^{K*}} (\|\mathcal{A}\|, t)$ [Baader, Peñaloza 2010]

Description Logics

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- Consistency of a TBox $\mathcal{T}$ (set of axioms): Is there a model of $\mathcal{T}$?
- Satisfiability of C w.r.t. $\mathcal{T}$: Is there a model $\mathcal{I}$ of $\mathcal{T}$ with $C^{\mathcal{I}} \neq \emptyset$?
- Subsumption $C \sqsubseteq_{\mathcal{T}} D$: Does $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ hold in all models $\mathcal{I}$ of $\mathcal{T}$?

Tree Models

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  ↳ identifying the axioms of $\mathcal{T}$ that are responsible for a contradiction
- $C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is unsatisfiable
- Inclusion test is useful for non-standard inferences

Inclusion and Complementation

Given two automata $\mathcal{A}$, $\mathcal{A}'$,
does $L(\mathcal{A}') \subseteq L(\mathcal{A})$ hold?

Given an automaton $\mathcal{A}$,
construct an automaton $\overline{\mathcal{A}}$ with $L(\overline{\mathcal{A}}) = \overline{L(\mathcal{A})}$.

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[Buhrke, Lescow, Vöge 1996; Kupferman, Vardi 1998; Vardi, Wilke 2008]:
Inclusion is in EXPTIME for parity automata

[Seidl 1989]:
Inclusion is EXPTIME-hard for automata on finite trees

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Modify algorithms for unweighted complementation for weighted automata:

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- Translation of the constructions and proofs to finite De Morgan lattices:
  - from 2^Q to S^Q
  - from $\wedge$ to $\otimes$ and $\vee$ to $\oplus$
  - from $\forall$ to $\bigotimes$ and $\exists$ to $\bigoplus$
  - from $q \in I$ to $\mathrm{in}(q)$ and $(\dots) \in \Delta$ to $\mathrm{wt}(\dots)$
  - from $x \Rightarrow y$ to $\overline{x} \oplus y$ or $x \leq y$

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Example:

"If r is a successful run of $\mathcal{A}$ on t and r_c is a successul run of $\overline{\mathcal{A}}$ on t, then all paths p of length m have a node $u \in p$ such that $r(u) \notin r_c(u)$."

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Only correct for Boolean lattices

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We need exponentially many inclusion tests between unweighted automata.

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Summary:

- black-box ($n2^m$) is faster and more general than this glass-box approach (2^{nm})
- optimizations of glass-box algorithm?

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Applications:

- lattice-weighted automata for axiom pinpointing
- automata-based reasoning in fuzzy description logics

Thank You

- Franz Baader and Rafael Peñaloza: Automata-based axiom pinpointing.
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