TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institute of Theoretical Computer Science Chair of Automata Theory

FUZZY ONTOLOGIES OVER LATTICES WITH T-NORMS

Stefan Borgwardt Rafael Peñaloza

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Outline

Introduction

An Ł-fuzzy Description Logic

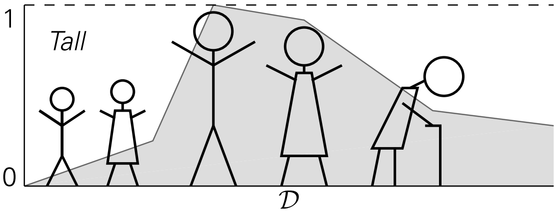
Satisfiability is Undecidable

Satisfiability is EXPTIME-Complete

Conclusions

Introduction

- Modeling of vague concepts
- Fuzzy sets $C : \mathcal{D} \to [0,1]$ instead of crisp sets $C \subseteq \mathcal{D}$ [Zadeh, 1965]



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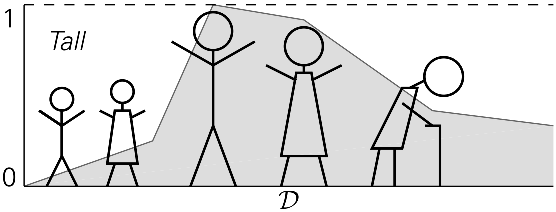
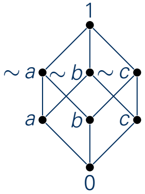
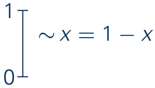


- Generalization to lattices and L-fuzzy sets $C : \mathcal{D} \to L$ [Goguen, 1967]
- $\mathcal{ALC}$ augmented with L-fuzzy semantics: $\mathcal{ALC}_L$

Lattices

complete, distributive lattice $L = (L, \vee, \wedge, 0, 1)$
with De Morgan negation $\sim : L \to L$





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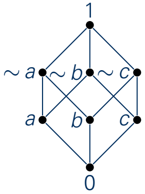

$\sim x = 1 - x$

additional operators:

- T-norm $\otimes : L \times L \to L$: associative, commutative, unit 1, monotone
- T-conorm (S-norm) $\ell \oplus m := \sim(\sim \ell \otimes \sim m)$
- R-implication $\ell \Rightarrow m := \bigvee\{x \mid \ell \otimes x \leq m\}$

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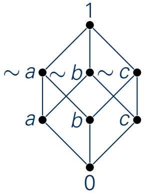

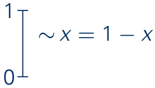

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$$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to L \qquad r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to L$$

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- C is strongly ℓ-satisfiable iff $C^{\mathcal{I}}(x) \geq \ell$ for some interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$

Satisfiability is Undecidable in $\mathcal{ALC}_L$

Lattice over $[0, 1] \cup \{-\infty, \infty\}$ with

$$\ell \otimes m := \begin{cases} \max\{\ell + m - 1, 0\} & \text{if } \ell, m \in [0, 1] \text{ and } \ell + m \neq 0, \\ -\infty & \text{if } \ell = m = 0, \text{ and} \\ \min\{\ell, m\} & \text{otherwise.} \end{cases}$$

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Over this lattice, the problem of deciding whether a concept C is (strongly) ℓ-satisfiable w.r.t. a general TBox $\mathcal{T}$ is undecidable.

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1) *n*-witnessed models:
   - for every $\exists r.C$ and $x \in \Delta^{\mathcal{I}}$ there are $x_1, \ldots, x_n \in \Delta^{\mathcal{I}}$ with

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   - (similary for $\forall r.C$)

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   For finite L, $\mathcal{ALC}_L$ has the *n*-witnessed model property for some $n \leq |L|$.

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2) C is ℓ-satisfiable $\iff \ell \leq$ best satisfiability degree of C

   best satisfiability degree of C = $\bigvee\{\ell \in L \mid C$ is strongly ℓ-satisfiable$\}$

Hintikka trees

$$x \otimes y = x \wedge y$$
$$x \oplus y = x \vee y$$
$$y \Rightarrow y = \sim x \vee y$$

$x \Rightarrow y$	f	m	t
f	t	t	t
m	m	m	t
t	f	m	t

Hintikka trees

- true
- maybe
- false

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$A \sqcap \forall r.B$ is maybe-satisfiable w.r.t. $\{\langle A \sqsubseteq \exists r.A \sqcup B, \mathit{true}\rangle\}$:

$A, B, \exists r.A, \exists r.A \sqcup B, \forall r.B, \underline{A \sqcap \forall r.B}$

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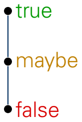

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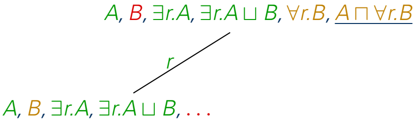

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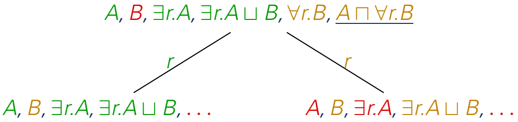

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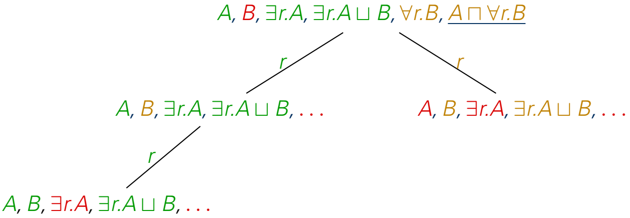

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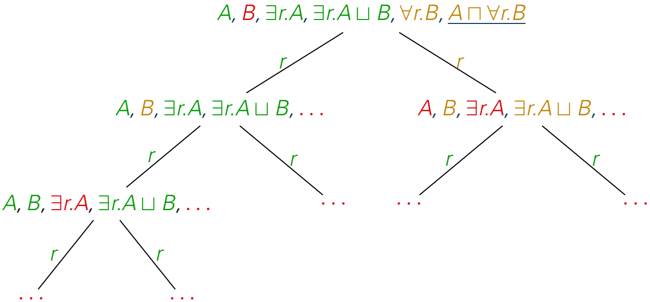

Hintikka trees

Hintikka set: $H : \mathrm{sub}(C, \mathcal{T}) \cup \{\rho\} \to L$ with

(i) $H(D \sqcap E) = H(D) \otimes H(E)$

(ii) $H(D \sqcup E) = H(D) \oplus H(E)$

(iii) $H(\neg D) = \sim H(D)$

(iv) $H(D) \Rightarrow H(E) \geq \ell$ for $\langle D \sqsubseteq E, \ell \rangle$ in $\mathcal{T}$

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Hintikka condition: for a branch $\begin{array}{ccc} & H_0 & \\ \swarrow & & \searrow \\ H_1 & \cdots & H_k \end{array}$:

(i) $H_0(\exists r.D) = H_{\varphi(\exists r.D)}(\rho) \otimes H_{\varphi(\exists r.D)}(D)$ and
$H_0(\exists r.D) \geq H_j(\rho) \otimes H_j(D)$ for every $j \neq \varphi(\exists r.D)$

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(ii) (similarly for $\forall r.D$)

- these conditions can be checked locally
- these trees can be recognized by an automaton over infinite trees
- complexity: exponential time
- computation of the best satisfiability degree without additional effort

Conclusions

Summary

- General framework for fuzzy reasoning
- Satisfiability undecidable for arbitrary lattices
- ExpTime-complete for finite lattices

Future Work

- Identify other decidable subclasses
- Reason also with ABoxes
- Find applications and identify useful semantics
- Implement efficient automata-based reasoners

Thank you

Fernando Bobillo, Félix Bou, and Umberto Straccia. On the failure of the finite model property in some fuzzy description logics. *Fuzzy Sets and Systems*, 172(1):1–12, 2011.

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