TECHNISCHE UNIVERSITÄT DRESDEN

cfaed CENTER FOR ADVANCING ELECTRONICS DRESDEN

DRESDEN concept

Institute of Theoretical Computer Science Chair of Automata Theory

GÖDEL $\mathcal{FL}_0$ WITH GREATEST FIXED-POINT SEMANTICS

Stefan Borgwardt José A. Leyva Galano Rafael Peñaloza

Wien, July 18th, 2014

Gödel Fuzzy Logics

$\mathsf{PanicAttackVictim} \sqsubseteq \forall \mathsf{extremity.Numb}$

Gödel Fuzzy Logics

$$\text{PanicAttackVictim} \sqsubseteq \forall \text{extremity}.\text{Numb}$$

- fuzzy set: $\text{Numb} \colon \Delta \to [0, 1]$ (Zadeh 1965)

Gödel Fuzzy Logics

$$\mathsf{PanicAttackVictim} \sqsubseteq \forall \mathsf{extremity}.\mathsf{Numb} \geq 0.4$$

- fuzzy set: Numb: $\Delta \to [0, 1]$ (Zadeh 1965)
- vagueness in the axioms

Gödel Fuzzy Logics

$$\mathsf{PanicAttackVictim} \sqsubseteq \forall \mathsf{extremity.Numb} \geq 0.4$$

- fuzzy set: Numb: $\Delta \to [0, 1]$ (Zadeh 1965)
- vagueness in the axioms
- combination of vague predicates via Gödel t-norm: (Gödel 1932)

$$[0, 1] \times [0, 1] \to [0, 1] \colon (x, y) \mapsto \min(x, y)$$

Gödel Fuzzy Logics

$$\mathsf{PanicAttackVictim} \sqsubseteq \forall \mathsf{extremity}.\mathsf{Numb} \geq 0.4$$

- fuzzy set: Numb : $\Delta \to [0, 1]$ (Zadeh 1965)
- vagueness in the axioms
- combination of vague predicates via Gödel t-norm: (Gödel 1932)

$$[0, 1] \times [0, 1] \to [0, 1] : (x, y) \mapsto \min(x, y)$$

- Gödel residuum for implication:

$$x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Gödel Fuzzy Logics

$$\mathsf{PanicAttackVictim} \sqsubseteq \forall \mathsf{extremity}.\mathsf{Numb} \geq 0.4$$

- fuzzy set: Numb: $\Delta \to [0, 1]$ (Zadeh 1965)
- vagueness in the axioms
- combination of vague predicates via Gödel t-norm: (Gödel 1932)

$$[0, 1] \times [0, 1] \to [0, 1] : (x, y) \mapsto \min(x, y)$$

- Gödel residuum for implication:

$$x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

$$x \leq y \text{ iff } (x \Rightarrow y) = 1 \qquad \min(x, z) \leq y \text{ iff } z \leq (x \Rightarrow y)$$

The Fuzzy DL G-$\mathcal{FL}_0$

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: $\mathsf{Happy}^\mathcal{I} \colon \Delta^\mathcal{I} \to [0, 1]$
- role names: $\mathsf{hasFriend}^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0, 1]$
- individual names: $\mathsf{stefan}^\mathcal{I} \in \Delta^\mathcal{I}$

The Fuzzy DL G-$\mathcal{FL}_0$

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: Happy$^\mathcal{I} \colon \Delta^\mathcal{I} \to [0, 1]$
- role names: hasFriend$^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0, 1]$
- individual names: stefan$^\mathcal{I} \in \Delta^\mathcal{I}$

Constructors:

- top $\top^\mathcal{I}(x) = 1$
- conjunction $(\mathsf{Happy} \sqcap \mathsf{Successful})^\mathcal{I}(x) = \min(\mathsf{Happy}^\mathcal{I}(x), \mathsf{Successful}^\mathcal{I}(x))$

The Fuzzy DL G-$\mathcal{FL}_0$

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: $\mathsf{Happy}^\mathcal{I} \colon \Delta^\mathcal{I} \to [0,1]$
- role names: $\mathsf{hasFriend}^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$
- individual names: $\mathsf{stefan}^\mathcal{I} \in \Delta^\mathcal{I}$

Constructors:

- top $\top^\mathcal{I}(x) = 1$
- conjunction $(\mathsf{Happy} \sqcap \mathsf{Successful})^\mathcal{I}(x) = \min(\mathsf{Happy}^\mathcal{I}(x), \mathsf{Successful}^\mathcal{I}(x))$
- value restriction

$$(\forall \mathsf{hasFriend}.\mathsf{Happy})^\mathcal{I}(x) = \inf_{y \in \Delta^\mathcal{I}} \mathsf{hasFriend}^\mathcal{I}(x,y) \Rightarrow \mathsf{Happy}^\mathcal{I}(y)$$

The Fuzzy DL G-$\mathcal{FL}_0$

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: $\mathsf{Happy}^\mathcal{I} \colon \Delta^\mathcal{I} \to [0,1]$
- role names: $\mathsf{hasFriend}^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$
- individual names: $\mathsf{stefan}^\mathcal{I} \in \Delta^\mathcal{I}$

Constructors:

- top $\top^\mathcal{I}(x) = 1$
- conjunction $(\mathsf{Happy} \sqcap \mathsf{Successful})^\mathcal{I}(x) = \min(\mathsf{Happy}^\mathcal{I}(x), \mathsf{Successful}^\mathcal{I}(x))$
- value restriction

$$(\forall\mathsf{hasFriend}.\mathsf{Happy})^\mathcal{I}(x) = \inf_{y \in \Delta^\mathcal{I}} \mathsf{hasFriend}^\mathcal{I}(x,y) \Rightarrow \mathsf{Happy}^\mathcal{I}(y)$$

Primitive concept definitions: $\langle \mathsf{Happy} \sqsubseteq \forall\mathsf{hasFriend}.\mathsf{Happy} \geq 0.8 \rangle$

The Fuzzy DL G-$\mathcal{FL}_0$

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: $\mathsf{Happy}^\mathcal{I} \colon \Delta^\mathcal{I} \to [0,1]$
- role names: $\mathsf{hasFriend}^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$
- individual names: $\mathsf{stefan}^\mathcal{I} \in \Delta^\mathcal{I}$

Constructors:

- top $\top^\mathcal{I}(x) = 1$
- conjunction $(\mathsf{Happy} \sqcap \mathsf{Successful})^\mathcal{I}(x) = \min(\mathsf{Happy}^\mathcal{I}(x), \mathsf{Successful}^\mathcal{I}(x))$
- value restriction

$$(\forall \mathsf{hasFriend}.\mathsf{Happy})^\mathcal{I}(x) = \inf_{y \in \Delta^\mathcal{I}} \mathsf{hasFriend}^\mathcal{I}(x,y) \Rightarrow \mathsf{Happy}^\mathcal{I}(y)$$

Primitive concept definitions: $\langle \mathsf{Happy} \sqsubseteq \forall \mathsf{hasFriend}.\mathsf{Happy} \geq 0.8 \rangle$

$$\mathsf{Happy}^\mathcal{I}(x) \Rightarrow (\forall \mathsf{hasFriend}.\mathsf{Happy})^\mathcal{I}(x) \geq 0.8$$

$$\min(0.8, \mathsf{Happy}^\mathcal{I}(x)) \leq (\forall \mathsf{hasFriend}.\mathsf{Happy})^\mathcal{I}(x)$$

Greatest Fixed-Point Semantics

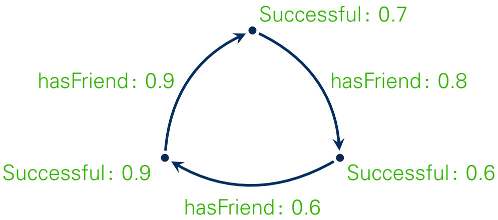
VIP: "successful people with only successful friends"

$$\langle \mathsf{VIP} \sqsubseteq \mathsf{Successful} \sqcap \forall \mathsf{hasFriend}.\mathsf{VIP} \geq 1 \rangle$$

Greatest Fixed-Point Semantics

VIP: "successful people with only successful friends"

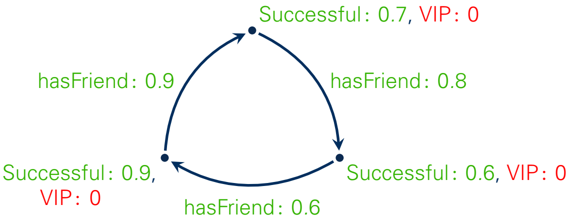
$\langle \text{VIP} \sqsubseteq \text{Successful} \sqcap \forall \text{hasFriend}.\text{VIP} \geq 1 \rangle$



Greatest Fixed-Point Semantics

VIP: "successful people with only successful friends"

$\langle \text{VIP} \sqsubseteq \text{Successful} \sqcap \forall \text{hasFriend.VIP} \geq 1 \rangle$

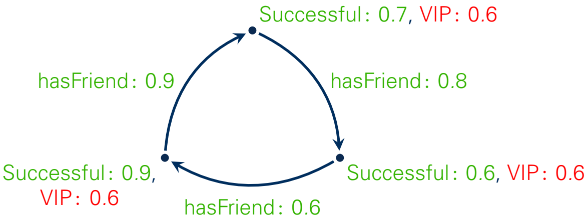


. . . is a model.

Greatest Fixed-Point Semantics

VIP: "successful people with only successful friends"

$\langle \text{VIP} \sqsubseteq \text{Successful} \sqcap \forall \text{hasFriend}.\text{VIP} \geq 1 \rangle$

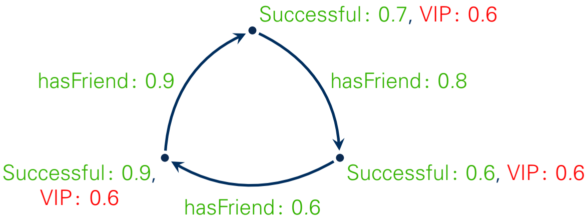


. . . is the greatest fixed-point model.

Greatest Fixed-Point Semantics

VIP: "successful people with only successful friends"

$\langle \mathsf{VIP} \sqsubseteq \mathsf{Successful} \sqcap \forall \mathsf{hasFriend}.\mathsf{VIP} \geq 1 \rangle$



. . . is the greatest fixed-point model.

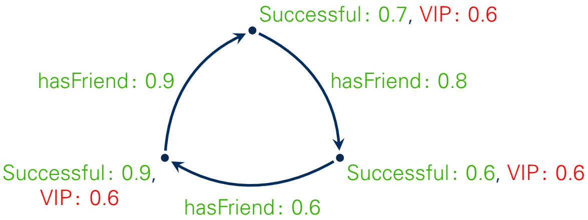
VIP has the largest possible extension:

$$\mathsf{VIP}^{\mathcal{I}}(x) = (\mathsf{Successful} \sqcap \forall \mathsf{hasFriend}.\mathsf{VIP})^{\mathcal{I}}(x)$$

Greatest Fixed-Point Semantics

VIP: "successful people with only successful friends"

$\langle \text{VIP} \sqsubseteq \text{Successful} \sqcap \forall \text{hasFriend}.\text{VIP} \geq 1 \rangle$



... is the greatest fixed-point model.

VIP has the largest possible extension:

$$\text{VIP}^{\mathcal{I}}(x) = (\text{Successful} \sqcap \forall \text{hasFriend}.\text{VIP})^{\mathcal{I}}(x)$$

$$\text{VIP}^{\mathcal{I}_{i+1}}(x) := (\text{Successful} \sqcap \forall \text{hasFriend}.\text{VIP})^{\mathcal{I}_i}(x)$$

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend

defined names: VIP

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend

defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ

extension function $f\colon \mathsf{N}_D^{\mathcal{T}} \to [0,1]^{\Delta}$

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend

defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ

extension function $f\colon \mathsf{N}_\mathsf{D}^{\mathcal{T}} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \rangle \in \mathcal{T}} C^{\mathcal{I}_{\mathcal{J},f_i}}(x)$$

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend

defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ

extension function $f\colon \mathsf{N}_\mathsf{D}^{\mathcal{T}} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}} (p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f_i}}(x))$$

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend

defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ

extension function $f \colon \mathsf{N}_\mathsf{D}^\mathcal{T} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p \rangle \in \mathcal{T}} \left(p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f_i}}(x)\right)$$

operator $T \colon f_i \mapsto f_{i+1}$ over $L := \left([0,1]^\Delta\right)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$:

- L is a complete lattice

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend — defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ — extension function $f\colon \mathsf{N}_\mathsf{D}^\mathcal{T} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}} \left(p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f_i}}(x)\right)$$

operator $T\colon f_i \mapsto f_{i+1}$ over $L := \left([0,1]^\Delta\right)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$:

- L is a complete lattice
- T is monotone $\rightsquigarrow$ $\mathrm{gfp}(T)$ exists (Tarski 1955)

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend — defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ — extension function $f\colon \mathsf{N}_\mathsf{D}^\mathcal{T} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}} \left(p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f_i}}(x)\right)$$

operator $T\colon f_i \mapsto f_{i+1}$ over $L := \left([0,1]^\Delta\right)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$:

- L is a complete lattice
- T is monotone $\rightsquigarrow$ gfp(T) exists (Tarski 1955)
- T is downward ω-continuous $\rightsquigarrow$ $\mathrm{gfp}(T) = \bigwedge_{i\geq 0} \left(T\right)^i(\mathbf{1})$ (Kleene 1952)

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend — defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ — extension function $f \colon \mathsf{N}_\mathsf{D}^\mathcal{T} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p \rangle \in \mathcal{T}} \left(p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f_i}}(x)\right)$$

operator $T \colon f_i \mapsto f_{i+1}$ over $L := \left([0,1]^\Delta\right)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$:

- L is a complete lattice
- T is monotone $\rightsquigarrow$ gfp(T) exists (Tarski 1955)
- T is downward ω-continuous $\rightsquigarrow$ $\mathrm{gfp}(T) = \bigwedge_{i \geq 0} (T)^i(\mathbf{1})$ (Kleene 1952)
- gfp(T) is a model of $\mathcal{T}$

Greatest Fixed-Point Semantics

primitive names: Successful, hasFriend   defined names: VIP

primitive interpretation $\mathcal{J}$ over Δ extension function $f\colon \mathsf{N}_\mathsf{D}^\mathcal{T} \to [0,1]^\Delta$

$$f_{i+1}(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p \rangle \in \mathcal{T}} \left(p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f_i}}(x)\right)$$

operator $T\colon f_i \mapsto f_{i+1}$ over $L := \left([0,1]^\Delta\right)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$:

- L is a complete lattice
- T is monotone $\rightsquigarrow$ gfp(T) exists   (Tarski 1955)
- T is downward ω-continuous $\rightsquigarrow$ $\text{gfp}(T) = \bigwedge_{i \geq 0} (T)^i(\mathbf{1})$ (Kleene 1952)
- gfp(T) is a model of $\mathcal{T}$

- gfp-subsumption: $\mathcal{T} \models_{\text{gfp}} \langle A \sqsubseteq B \geq p \rangle$
- best gfp-subsumption degree

Related Work

Gödel description logics:

- in G-$\mathcal{EL}^{++}$ subsumption w.r.t. GCIs is polynomial (Mailis et al. 2012)
- in G-$\mathcal{ALC}$ reasoning w.r.t. GCIs is ExpTime-complete (Borgwardt et al. 2014)
- in finite-valued G_n-$\mathcal{SROIQ}$ reasoning is decidable (Bobillo et al. 2009)

Related Work

Gödel description logics:

- in G-$\mathcal{EL}^{++}$ subsumption w.r.t. GCIs is polynomial (Mailis et al. 2012)
- in G-$\mathcal{ALC}$ reasoning w.r.t. GCIs is ExpTime-complete (Borgwardt et al. 2014)
- in finite-valued G_n-$\mathcal{SROIQ}$ reasoning is decidable (Bobillo et al. 2009)

fuzzy description logics using other t-norms:

- in $\otimes$-$\mathcal{ALC}$ reasoning w.r.t. GCIs is undecidable (Borgwardt and Peñaloza 2012; Cerami and Straccia 2013)
- in finite-valued $\otimes$-$\mathcal{SHOI}$ reasoning is decidable (Borgwardt 2014; Borgwardt and Peñaloza 2013)

Related Work

Gödel description logics:

- in G-$\mathcal{EL}^{++}$ subsumption w.r.t. GCIs is polynomial (Mailis et al. 2012)
- in G-$\mathcal{ALC}$ reasoning w.r.t. GCIs is ExpTime-complete (Borgwardt et al. 2014)
- in finite-valued G_n-$\mathcal{SROIQ}$ reasoning is decidable (Bobillo et al. 2009)

fuzzy description logics using other t-norms:

- in $\otimes$-$\mathcal{ALC}$ reasoning w.r.t. GCIs is undecidable (Borgwardt and Peñaloza 2012; Cerami and Straccia 2013)
- in finite-valued $\otimes$-$\mathcal{SHOI}$ reasoning is decidable (Borgwardt 2014; Borgwardt and Peñaloza 2013)

classical $\mathcal{FL}_0$:

- subsumption w.r.t. GCIs is ExpTime-hard (Baader et al. 2005)
- subsumption w.r.t. acyclic TBoxes is co-NP-complete (Nebel 1990)
- gfp-subsumption w.r.t. cyclic TBoxes is PSpace-complete (Baader 1990)

Related Work

Gödel description logics:

- in G-$\mathcal{EL}^{++}$ subsumption w.r.t. GCIs is polynomial (Mailis et al. 2012)
- in G-$\mathcal{ALC}$ reasoning w.r.t. GCIs is ExpTime-complete (Borgwardt et al. 2014)
- in finite-valued G$_n$-$\mathcal{SROIQ}$ reasoning is decidable (Bobillo et al. 2009)

fuzzy description logics using other t-norms:

- in $\otimes$-$\mathcal{ALC}$ reasoning w.r.t. GCIs is undecidable (Borgwardt and Peñaloza 2012; Cerami and Straccia 2013)
- in finite-valued $\otimes$-$\mathcal{SHOI}$ reasoning is decidable (Borgwardt 2014; Borgwardt and Peñaloza 2013)

classical $\mathcal{FL}_0$:

- subsumption w.r.t. GCIs is ExpTime-hard (Baader et al. 2005)
- subsumption w.r.t. acyclic TBoxes is co-NP-complete (Nebel 1990)
- gfp-subsumption w.r.t. cyclic TBoxes is PSpace-complete (Baader 1990)

gfp-subsumption in G-$\mathcal{FL}_0$ is co-NP-complete/PSpace-complete

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p \rangle$

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p \rangle$

$\langle \text{VIP} \sqsubseteq \text{Successful} \sqcap \forall \text{hasFriend}.\text{VIP} \geq 1 \rangle$

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p \rangle$

$\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathsf{paths}(\mathcal{A},w)} \mathsf{wt}(\pi)$

$\langle \mathsf{VIP} \sqsubseteq \mathsf{Successful} \sqcap \forall \mathsf{hasFriend}.\mathsf{VIP} \geq 1 \rangle$

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p \rangle$

$$\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathsf{paths}(\mathcal{A},w)} \mathsf{wt}(\pi)$$

$\langle \mathsf{VIP} \sqsubseteq \mathsf{Successful} \sqcap \forall \mathsf{hasFriend}.\mathsf{VIP} \geq 1 \rangle$

$$\|\mathcal{A}_{\mathsf{VIP},\mathsf{S}}\|(\mathsf{hasFriend}^n) = 1$$

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p \rangle$

$$\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathrm{paths}(\mathcal{A},w)} \mathrm{wt}(\pi)$$

$$A^{\mathcal{I}}(x) = \bigwedge_{C \in \mathsf{N}_\mathsf{P}^\mathcal{T}} \bigwedge_{w \in \mathsf{N}_\mathsf{R}^*} \left(\|\mathcal{A}_{A,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x) \right)$$

$\langle \mathrm{VIP} \sqsubseteq \mathrm{Successful} \sqcap \forall \mathrm{hasFriend}.\mathrm{VIP} \geq 1 \rangle$

$$\|\mathcal{A}_{\mathrm{VIP,S}}\|(\mathrm{hasFriend}^n) = 1$$

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p\rangle$

$$\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathrm{paths}(\mathcal{A},w)} \mathrm{wt}(\pi)$$

$$A^{\mathcal{I}}(x) = \bigwedge_{C \in \mathsf{N}_\mathsf{P}^{\mathcal{T}}} \bigwedge_{w \in \mathsf{N}_\mathsf{R}^*} (\|\mathcal{A}_{A,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x))$$

$\langle \mathrm{VIP} \sqsubseteq \mathrm{Successful} \sqcap \forall \mathrm{hasFriend}.\mathrm{VIP} \geq 1\rangle$

$$\|\mathcal{A}_{\mathrm{VIP},\mathrm{S}}\|(\mathrm{hasFriend}^n) = 1$$

$$\mathrm{VIP}^{\mathcal{I}}(x) = \bigwedge_{n \geq 0} (\forall \mathrm{hasFriend}^n.\mathrm{Successful})^{\mathcal{I}}(x)$$

Weighted Automata

normal form $\langle A \sqsubseteq \forall w.B \geq p \rangle$

$\langle \mathrm{VIP} \sqsubseteq \mathrm{Successful} \sqcap \forall \mathrm{hasFriend.VIP} \geq 1 \rangle$

$\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathrm{paths}(\mathcal{A},w)} \mathrm{wt}(\pi)$

$\|\mathcal{A}_{\mathrm{VIP,S}}\|(\mathrm{hasFriend}^n) = 1$

$$A^{\mathcal{I}}(x) = \bigwedge_{C \in \mathsf{N}_\mathsf{P}^{\mathcal{T}}} \bigwedge_{w \in \mathsf{N}_\mathsf{R}^*} \big(\|\mathcal{A}_{A,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x)\big)$$

$$\mathrm{VIP}^{\mathcal{I}}(x) = \bigwedge_{n \geq 0} (\forall \mathrm{hasFriend}^n.\mathrm{Successful})^{\mathcal{I}}(x)$$

characterization of gfp-subsumption:

$$\mathcal{T} \models_{\mathrm{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^{\mathcal{T}}$ and $w \in \mathsf{N}_\mathsf{R}^*$

Automata Inclusion

$$\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^{\mathcal{T}}$ and $w \in \mathsf{N}_\mathsf{R}^*$

Automata Inclusion

$$\mathcal{T} \models_{\mathrm{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$

$$\text{iff } L\big((\mathcal{A}_{B,C})_{\geq p} \big) \subseteq L\big((\mathcal{A}_{A,C})_{\geq p} \big)$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$

Automata Inclusion

$$\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$

$$\text{iff } L\big((\mathcal{A}_{B,C})_{\geq p'}\big) \subseteq L\big((\mathcal{A}_{A,C})_{\geq p'}\big)$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $p' \leq p$

Automata Inclusion

$$\mathcal{T} \models_{\text{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$

$$\text{iff } L\big((\mathcal{A}_{B,C})_{\geq p'}\big) \subseteq L\big((\mathcal{A}_{A,C})_{\geq p'}\big)$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $p' \leq p$

- polynomially many inclusion tests between NFA

Automata Inclusion

$$\mathcal{T} \models_{\text{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$

$$\text{iff } L\big((\mathcal{A}_{B,C})_{\geq p'}\big) \subseteq L\big((\mathcal{A}_{A,C})_{\geq p'}\big)$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $p' \leq p$

- polynomially many inclusion tests between NFA
- PSPACE-complete for cyclic TBoxes
- co-NP-complete for acyclic TBoxes (Garey and Johnson 1979)

Automata Inclusion

$$\mathcal{T} \models_{\text{gfp}} \langle A \sqsubseteq B \geq p \rangle \text{ iff } \big(\|\mathcal{A}_{B,C}\|(w) \Rightarrow \|\mathcal{A}_{A,C}\|(w)\big) \geq p$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$

$$\text{iff } L\big((\mathcal{A}_{B,C})_{\geq p'}\big) \subseteq L\big((\mathcal{A}_{A,C})_{\geq p'}\big)$$

for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $p' \leq p$

- polynomially many inclusion tests between NFA
- PSPACE-complete for cyclic TBoxes
- co-NP-complete for acyclic TBoxes (Garey and Johnson 1979)
- similar for computing best subsumption degrees

Conclusions

Summary:

- gfp-subsumption in G-$\mathcal{FL}_0$ is co-NP-complete/PSPACE-complete
- similar results for other Gödel fuzzy DLs

Conclusions

Summary:

- gfp-subsumption in G-$\mathcal{FL}_0$ is co-NP-complete/PSPACE-complete
- similar results for other Gödel fuzzy DLs

Future Work:

- least fixed-point semantics
- semantics based on other t-norms

Conclusions

Summary:

- gfp-subsumption in G-$\mathcal{FL}_0$ is co-NP-complete/PSPACE-complete
- similar results for other Gödel fuzzy DLs

Future Work:

- least fixed-point semantics
- semantics based on other t-norms

Thank you

References I

- Baader, Franz (1990). "Terminological cycles in KL-ONE-based knowledge representation languages". In: *Proc. AAAI'90*, pages 621–626.
- Baader, Franz, Sebastian Brandt, and Carsten Lutz (2005). "Pushing the $\mathcal{EL}$ Envelope". In: *Proc. IJCAI'05*, pages 364–369.
- Bobillo, Fernando, Miguel Delgado, Juan Gómez-Romero, and Umberto Straccia (2009). "Fuzzy Description Logics under Gödel Semantics". In: *Int. J. Approx. Reason.* 50.3, pages 494–514.
- Borgwardt, Stefan (2014). "Fuzzy DLs over Finite Lattices with Nominals". In: *Proc. DL'14*. Volume 1193. CEUR-WS, pages 58–70.
- Borgwardt, Stefan, Felix Distel, and Rafael Peñaloza (2014). "Decidable Gödel Description Logics without the Finitely-Valued Model Property". In: *Proc. KR'14*. To appear.
- Borgwardt, Stefan and Rafael Peñaloza (2012). "Undecidability of Fuzzy Description Logics". In: *Proc. KR'12*, pages 232–242.
- — (2013). "The Complexity of Lattice-Based Fuzzy Description Logics". In: *J. Data Semant.* 2.1, pages 1–19.
- Cerami, Marco and Umberto Straccia (2013). "On the (Un)decidability of Fuzzy Description Logics under Łukasiewicz t-norm". In: *Inform. Sciences* 227, pages 1–21.
- Garey, Michael R. and David S. Johnson (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*.
- Gödel, Kurt (1932). "Zum intuitionistischen Aussagenkalkül". In: *Anzeiger der Akademie der Wissenschaften in Wien* 69. Reprinted in Gödel 1986, pages 286–295.
- Kleene, Stephen Cole (1952). *Introduction to Metamathematics*.

References II

- Mailis, Theofilos, Giorgos Stoilos, Nikolaos Simou, Giorgos B. Stamou, and Stefanos Kollias (2012). “Tractable Reasoning with Vague Knowledge using Fuzzy $\mathcal{EL}^{++}$”. In: *J. Intell. Inf. Syst.* 39.2, pages 399–440.
- Nebel, Bernhard (1990). “Terminological Reasoning is Inherently Intractable”. In: *Artif. Intell.* 43.2, pages 235–249.
- Tarski, Alfred (1955). “A Lattice-Theoretical Fixpoint Theorem and its Applications”. In: *Pac. J. Math.* 5.2, pages 285–309.
- Zadeh, Lotfi A. (1965). “Fuzzy Sets”. In: *Inform. Control* 8.3, pages 338–353.