TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institute of Theoretical Computer Science Chair of Automata Theory

FUZZY DLS OVER FINITE LATTICES WITH NOMINALS

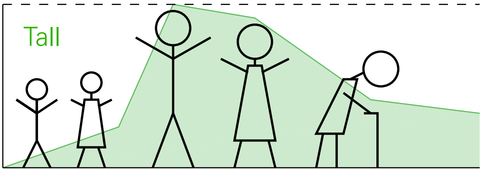
Stefan Borgwardt

Wien, July 18th, 2014

Introduction

Fuzzy logics:

- fuzzy sets $C\colon \Delta \to [0,1]$ replace crisp sets $C \subseteq \Delta$ (Zadeh 1965)



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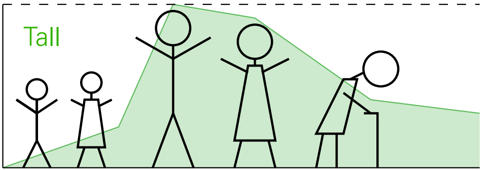

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- mathematical fuzzy logic with t-norms (Hájek 2001)

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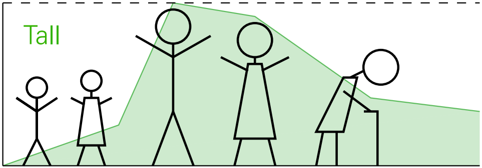

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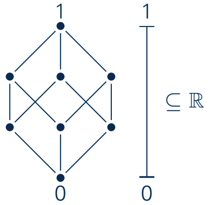
Fuzzy description logics:

- Zadeh semantics (Straccia 2001)
- lattice-based semantics (Straccia 2004b)
- t-norm-based semantics (Hájek 2005)

Complete Residuated De Morgan Lattices

complete distributive lattice $(L, \vee, \wedge, 0, 1)$:



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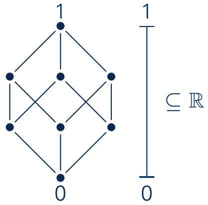

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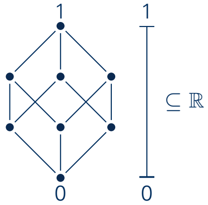

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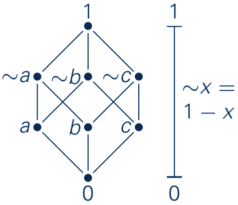

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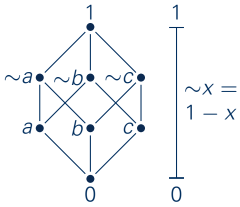

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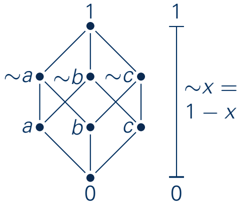

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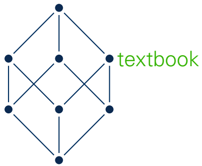

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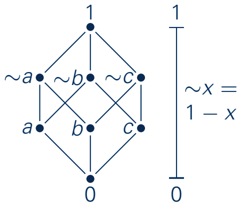

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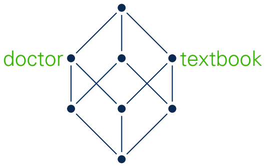

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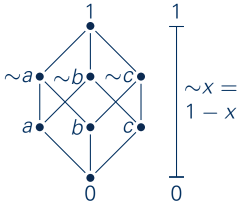

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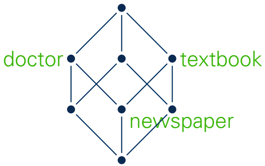

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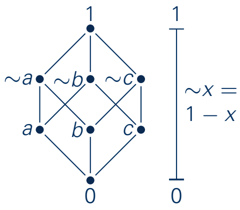

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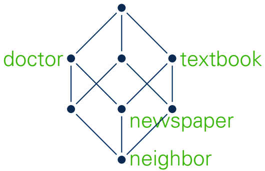

L-$\mathcal{ISCHOI}$

$\mathsf{Tall}^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to L$ $\mathsf{hasFriend}^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to L$ $\mathsf{stefan}^{\mathcal{I}} \in \Delta^{\mathcal{I}}$

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$$\overline{p}^{\mathcal{I}}(x) = p \qquad \{\mathsf{stefan}\}^{\mathcal{I}}(x) = \begin{cases} 1 & \text{if } \mathsf{stefan}^{\mathcal{I}} = x \\ 0 & \text{otherwise} \end{cases} \qquad (\neg\mathsf{Tall})^{\mathcal{I}}(x) = {\sim}\mathsf{Tall}^{\mathcal{I}}(x)$$

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\end{array}$$

subsumption: $\mathcal{O} \models \langle \mathsf{Successful} \sqsubseteq \neg\mathsf{Happy} \geq p \rangle$?

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iff $\mathcal{O} \cup \{\langle a{:}\mathsf{Successful} \to \neg\mathsf{Happy} < p\rangle\}$ is inconsistent

Related Work

Consistency is ...

- undecidable in small extensions of L-$\mathcal{EL}$ over many infinite lattices L
  (Baader and Peñaloza 2011; Cerami and Straccia 2013)

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- PSpace/ExpTime-complete in L-$\mathcal{ISCHI}$ over finite lattices L
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  ↳ extension to L-$\mathcal{ISCHOI}$:
  - construction for nominals
  - PSpace results for two sublogics

Pre-completions

$x \otimes y$	f	h	t
f	f	f	f
h	f	f	h
t	f	h	t

$x \Rightarrow y$	f	h	t
f	t	t	t
h	h	t	t
t	f	h	t

Pre-completions

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- partition of the individual names

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Pre-completion:

- partition of the individual names
- Hintikka functions: $\text{sub}(\mathcal{O}) \to L$
  H_{stefan}: Happy, $\exists$hasFriend.{rafael}, $\forall$hasFriend.Happy, {stefan}
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- role connections:
  $R_{\text{hasFriend}}$: (stefan, rafael), (rafael, stefan)

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- for ExpTime and PSpace upper bounds

Hintikka trees

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$$\langle \mathsf{Happy} \sqsubseteq \forall \mathsf{hasFriend}.\mathsf{Happy} \geq \mathsf{t} \rangle$$

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h	h	t	t
t	f	h	t

$\langle \text{Happy} \sqsubseteq \forall \text{hasFriend}.\text{Happy} \geq \text{t} \rangle$

$\text{hasFriend} \sqsubseteq \text{hasFriend}^{-}$

H_{stefan} : Happy, $\exists$hasFriend.{rafael}, $\forall$hasFriend.Happy, {stefan}

H_{rafael} : Happy, $\forall$hasFriend.Happy, {rafael}

$R_{\text{hasFriend}}$: (stefan, rafael), (rafael, stefan)

Hintikka trees:

Hintikka trees

$x \otimes y$	f	h	t
f	f	f	f
h	f	f	h
t	f	h	t

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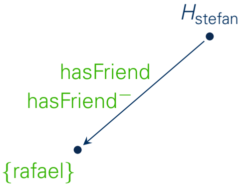

Hintikka trees

$x \otimes y$	f	h	t
f	f	f	f
h	f	f	h
t	f	h	t

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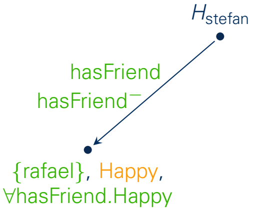

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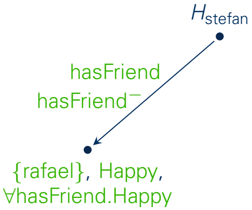

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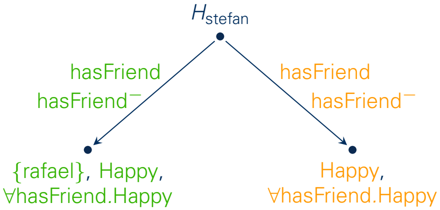

Hintikka trees

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f	f	f	f
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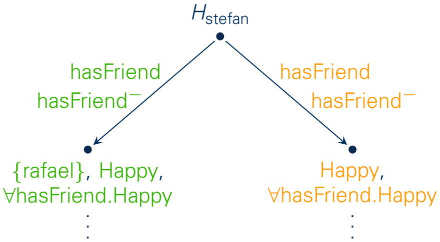

PSPACE on-the-fly constructions

Looping tree automata using Hintikka functions as states $\rightsquigarrow$ in EXPTIME

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Looping tree automata using Hintikka functions as states $\rightsquigarrow$ in ExpTime

PSpace on-the-fly constructions: (Baader, Hladik, and Peñaloza 2008)

- polynomial arity
- states of polynomial size
- polynomial initial and transition conditions
- polynomial blocking

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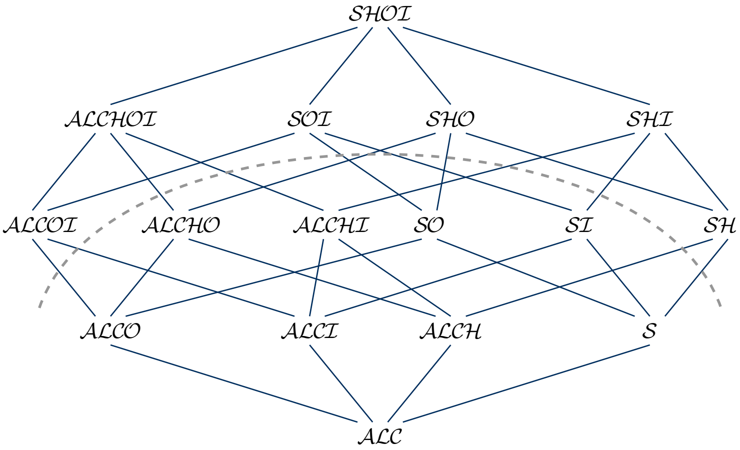
- polynomial arity
- states of polynomial size
- polynomial initial and transition conditions
- polynomial blocking
- can be achieved for L-$\mathcal{IALCHO}$ and L-$\mathcal{ISCO}_c$ with acyclic TBoxes $\rightsquigarrow$ in PSPACE

PSPACE/EXPTIME boundary



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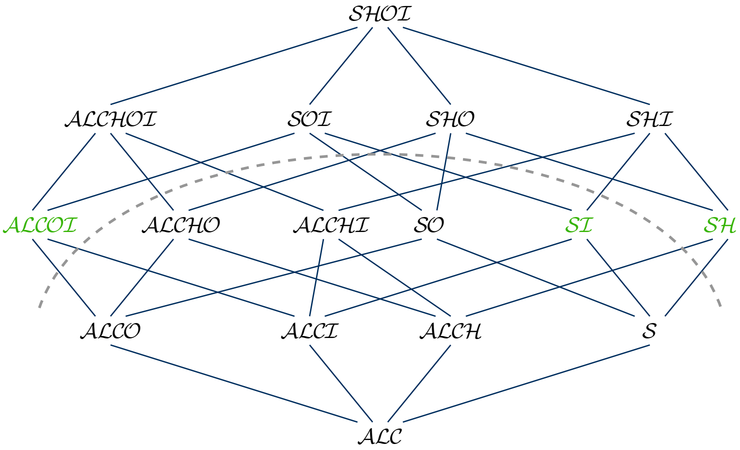

(Horrocks 1997; Horrocks, Sattler, and Tobies 2000; Tobies 2000)

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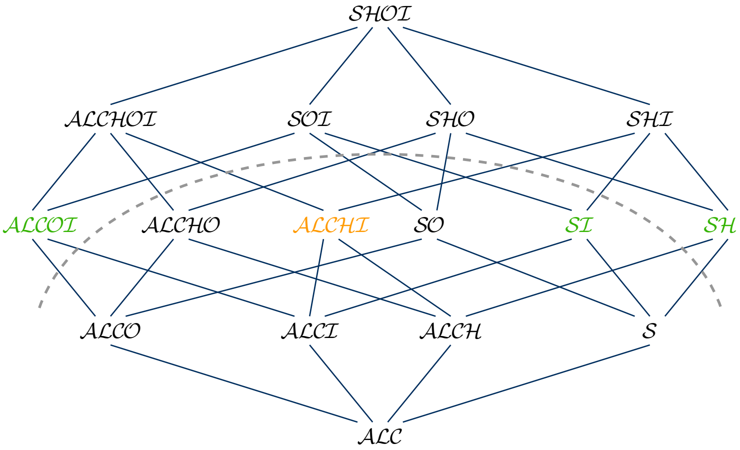

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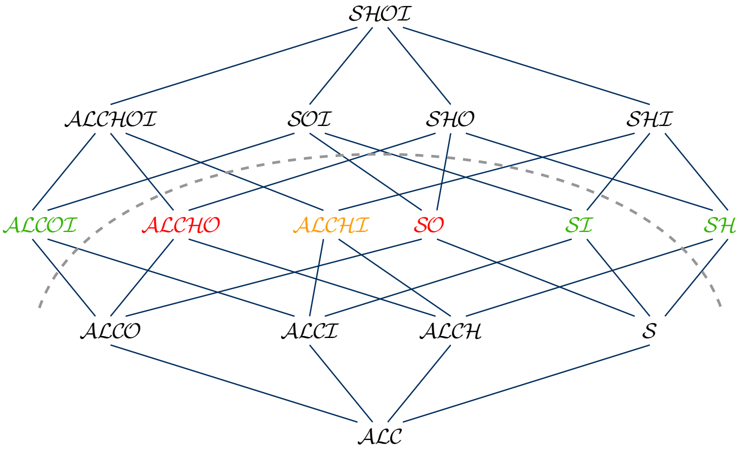
(Borgwardt and Peñaloza 2013)

PSpace/ExpTime boundary



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Conclusions

Summary:

- General finite-valued fuzzy semantics for $\mathcal{SHOI}$
- Same complexity as classical reasoning
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Thank you

References I

- Baader, Franz, Jan Hladik, and Rafael Peñaloza (2008). "Automata Can Show PSPACE Results for Description Logics". In: *Inform. Comput.* 206.9-10, pages 1045–1056.
- Baader, Franz and Rafael Peñaloza (2011). "On the Undecidability of Fuzzy Description Logics with GCIs and Product T-norm". In: *Proc. FroCoS'11*. Volume 6989. LNCS, pages 55–70.
- Bobillo, Fernando, Miguel Delgado, Juan Gómez-Romero, and Umberto Straccia (2012). "Joining Gödel and Zadeh Fuzzy Logics in Fuzzy Description Logics". In: *Int. J. Uncertain. Fuzz.* 20.4, pages 475–508.
- Borgwardt, Stefan (2014). "Fuzzy Description Logics with General Concept Inclusions". PhD thesis. Technische Universität Dresden.
- Borgwardt, Stefan, Felix Distel, and Rafael Peñaloza (2012). "How Fuzzy is my Fuzzy Description Logic?" In: *Proc. IJCAR'12*. Volume 7364. LNAI, pages 82–96.
- Borgwardt, Stefan and Rafael Peñaloza (2013). "The Complexity of Lattice-Based Fuzzy Description Logics". In: *J. Data Semant.* 2.1, pages 1–19.
- Cerami, Marco and Umberto Straccia (2013). "On the (Un)decidability of Fuzzy Description Logics under Łukasiewicz t-norm". In: *Inform. Sciences* 227, pages 1–21.
- Goguen, Joseph A. (1967). "L-Fuzzy Sets". In: *J. Math. Anal. Appl.* 18.1, pages 145–174.
- Hájek, Petr (2001). *Metamathematics of Fuzzy Logic (Trends in Logic)*.
- — (2005). "Making Fuzzy Description Logic more General". In: *Fuzzy Set. Syst.* 154.1, pages 1–15.
- Horrocks, Ian (1997). "Optimising Tableaux Decision Procedures for Description Logics". PhD thesis. University of Manchester.

References II

- Horrocks, Ian, Ulrike Sattler, and Stephan Tobies (2000). "Practical Reasoning for Very Expressive Description Logics". In: *L. J. IGPL* 8.3, pages 239–263.
- Straccia, Umberto (2001). "Reasoning within Fuzzy Description Logics". In: *J. Artif. Intell. Res.* 14, pages 137–166.
- — (2004a). "Transforming Fuzzy Description Logics into Classical Description Logics". In: *Proc. JELIA'04*. Volume 3229. LNCS, pages 385–399.
- — (2004b). "Uncertainty in Description Logics: A Lattice-based Approach". In: *Proc. IPMU'04*, pages 251–258.
- Tobies, Stephan (2000). "The Complexity of Reasoning with Cardinality Restrictions and Nominals in Expressive Description Logics". In: *J. Artif. Intell. Res.* 12, pages 199–217.
- Zadeh, Lotfi A. (1965). "Fuzzy Sets". In: *Inform. Control* 8.3, pages 338–353.