TECHNISCHE UNIVERSITÄT DRESDEN

CENTER FOR PERSPICUOUS COMPUTING

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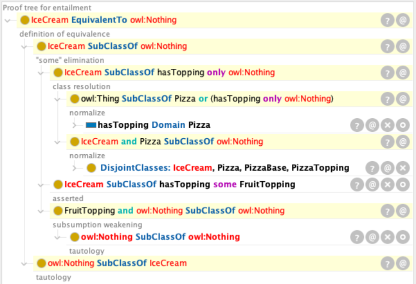
Technische Universität Dresden

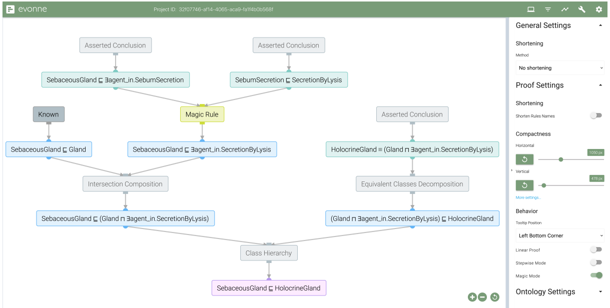
The Shape of $\mathcal{EL}$ Proofs: A Tale of Three Calculi

DL'25, Opole, Poland, September 3rd, 2025

How to explain description logic entailments?



Evee



Evonne

Why is this useful?

$$\left\{\begin{array}{cc} \mathsf{hasPaperAt} \sqsubseteq \mathsf{contributesTo} & \mathsf{DL25Author} \sqsubseteq \exists \mathsf{hasWorkshopPaperAt}.\mathsf{DL25} \\ \exists \mathsf{contributesTo}.\mathsf{DL} \sqsubseteq \mathsf{DLCommunityMember} & \mathsf{DL25FullPaperAuthor} \sqsubseteq \mathsf{DL25Author} \\ \mathsf{hasWorkshopPaperAt} \sqsubseteq \mathsf{hasPaperAt} & \mathsf{DL25} \sqsubseteq \mathsf{DL} \end{array}\right\}$$

$$\models \quad \mathsf{DL25FullPaperAuthor} \sqsubseteq \mathsf{DLCommunityMember}$$

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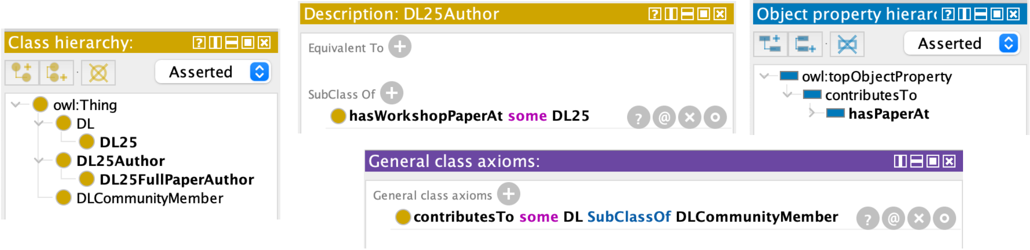

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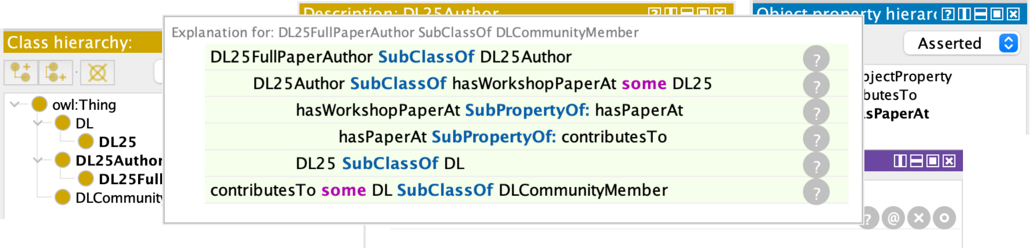

How does reasoning in $\mathcal{EL}$ work?

Pushing the $\mathcal{EL}$ Envelope

Franz Baader and **Sebastian Brandt** and **Carsten Lutz**
Intitute for Theoretical Computer Science
TU Dresden, Germany
lastname@tcs.inf.tu-dresden.de

ENVELOPE ($\mathcal{EL}^{++}$), 2005

CR1 If $C' \in S(C)$, $C' \sqsubseteq D \in \mathcal{C}$, and $D \notin S(C)$
then $S(C) := S(C) \cup \{D\}$

CR4 If $(C, D) \in R(r)$, $D' \in S(D)$, $\exists r.D' \sqsubseteq E \in \mathcal{C}$,
and $E \notin S(C)$
then $S(C) := S(C) \cup \{E\}$

How does reasoning in $\mathcal{EL}$ work?

The Incredible ELK

From Polynomial Procedures to Efficient Reasoning with $\mathcal{EL}$ Ontologies

Yevgeny Kazakov · Markus Krötzsch · František Simančík

ELK ($\mathcal{EL}^{+}_{\bot}$), 2014

$$\mathsf{R}_{\sqsubseteq}\ \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$$

$$\mathsf{R}^{+}_{\exists}\ \frac{E \xrightarrow{R} C \quad C \sqsubseteq D}{E \sqsubseteq \exists S.D} : \begin{array}{l} R \sqsubseteq^{*}_{\mathcal{O}} S \\ \exists S.D \text{ occurs negatively in } \mathcal{O} \end{array}$$

ENVELOPE ($\mathcal{EL}^{++}$), 2005

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Textbook ($\mathcal{EL}$), 2017

$$\text{CR3}\quad \frac{A_1 \sqsubseteq A_2 \quad A_2 \sqsubseteq A_3}{A_1 \sqsubseteq A_3}$$

$$\text{CR5}\quad \frac{A \sqsubseteq \exists r.A_1 \quad A_1 \sqsubseteq B_1 \quad \exists r.B_1 \sqsubseteq B}{A \sqsubseteq B}$$

How does reasoning in $\mathcal{EL}$ work?

To compare them, we use $\mathcal{ELH}_\bot$

Envelope ($\mathcal{EL}^{++}$), 2005

CR1 If $C' \in S(C)$, $C' \sqsubseteq D \in \mathcal{C}$, and $D \notin S(C)$ then $S(C) := S(C) \cup \{D\}$

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Elk ($\mathcal{EL}^+_\bot$), 2014

$$\mathsf{R}_\sqsubseteq \; \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$$

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What do the proofs look like?

$\mathrm{DL25FPA} \sqsubseteq \mathrm{DL25A}$ $\quad$ $\mathrm{DL25A} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}$ $\quad$ $\mathrm{hasWP} \sqsubseteq \mathrm{hasP}$ $\quad$ $\mathrm{hasP} \sqsubseteq \mathrm{cT}$

$\mathrm{DL25FPA} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}$ $\quad$ $\mathrm{DL25} \sqsubseteq \mathrm{DL}$ $\quad$ $\mathrm{hasWP} \sqsubseteq \mathrm{cT}$ $\quad$ $\exists \mathrm{cT}.\mathrm{DL} \sqsubseteq \mathrm{DLCM}$

TEXTBOOK

$\mathrm{DL25FPA} \sqsubseteq \mathrm{DLCM}$

What do the proofs look like?

TEXTBOOK

$$\dfrac{\dfrac{\mathrm{DL25FPA} \sqsubseteq \mathrm{DL25A} \quad \mathrm{DL25A} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}}{\mathrm{DL25FPA} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}} \quad \mathrm{DL25} \sqsubseteq \mathrm{DL} \quad \dfrac{\mathrm{hasWP} \sqsubseteq \mathrm{hasP} \quad \mathrm{hasP} \sqsubseteq \mathrm{cT}}{\mathrm{hasWP} \sqsubseteq \mathrm{cT}} \quad \exists \mathrm{cT}.\mathrm{DL} \sqsubseteq \mathrm{DLCM}}{\mathrm{DL25FPA} \sqsubseteq \mathrm{DLCM}}$$

ELK

$$\dfrac{\dfrac{\dfrac{\mathrm{DL25FPA} \sqsubseteq \mathrm{DL25A} \quad \mathrm{DL25A} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}}{\mathrm{DL25FPA} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}} \quad \mathrm{DL25} \sqsubseteq \mathrm{DL} \quad \dfrac{\mathrm{hasWP} \sqsubseteq \mathrm{hasP} \quad \mathrm{hasP} \sqsubseteq \mathrm{cT}}{\mathrm{hasWP} \sqsubseteq \mathrm{cT}}}{\mathrm{DL25FPA} \sqsubseteq \exists \mathrm{cT}.\mathrm{DL}} \quad \exists \mathrm{cT}.\mathrm{DL} \sqsubseteq \mathrm{DLCM}}{\mathrm{DL25FPA} \sqsubseteq \mathrm{DLCM}}$$

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Textbook $\quad \mathrm{DL25FPA} \sqsubseteq \mathrm{DLCM}$

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Elk $\quad \mathrm{DL25FPA} \sqsubseteq \exists \mathrm{hasWP}.\mathrm{DL25}$ $\quad \mathrm{DL25} \sqsubseteq \mathrm{DL}$ $\quad \mathrm{hasWP} \sqsubseteq \mathrm{cT}$

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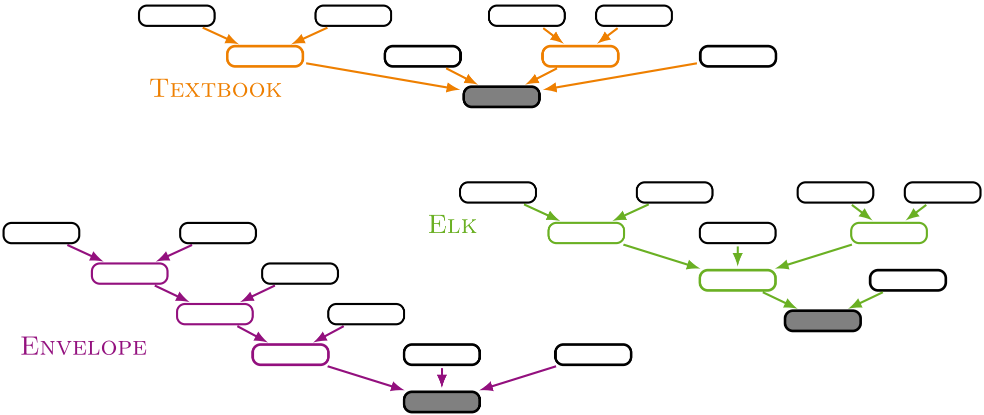
Envelope $\quad \mathrm{DL25FPA} \sqsubseteq \exists \mathrm{cT}.\mathrm{DL25}$ $\quad \mathrm{DL25} \sqsubseteq \mathrm{DL}$ $\quad \exists \mathrm{cT}.\mathrm{DL} \sqsubseteq \mathrm{DLCM}$

$\mathrm{DL25FPA} \sqsubseteq \mathrm{DLCM}$

What is the structure of the proofs?



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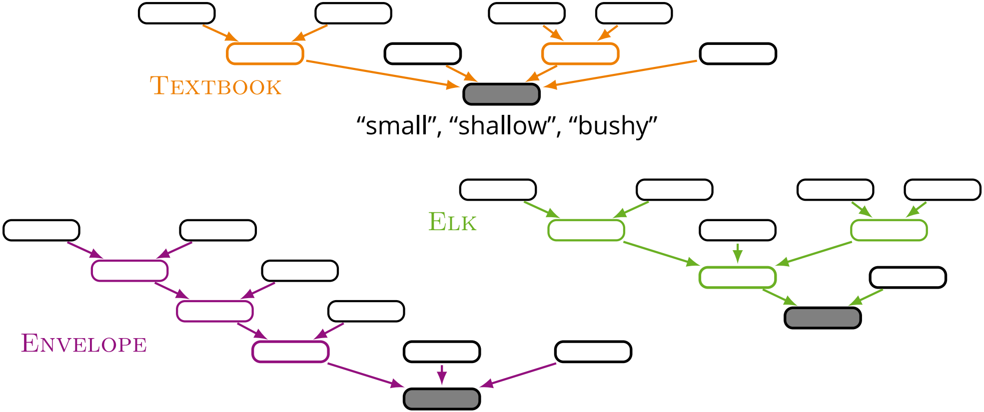

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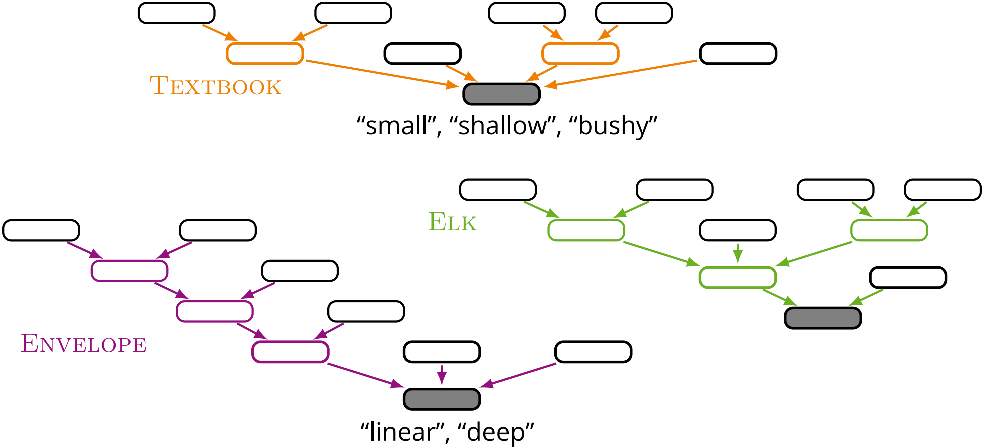
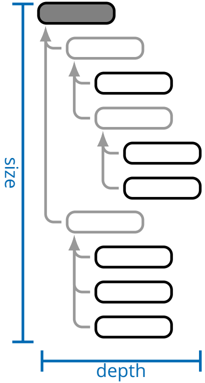


How can we display and measure proofs?



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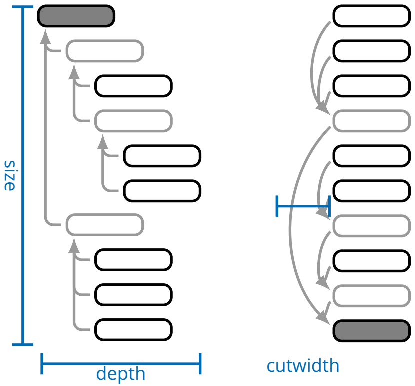

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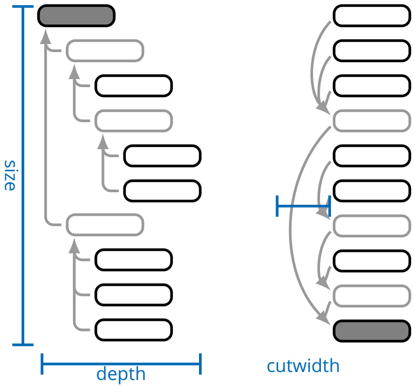
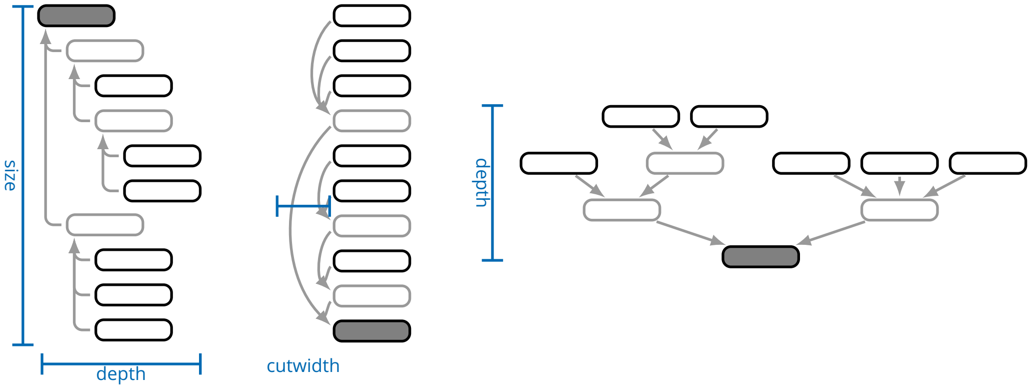


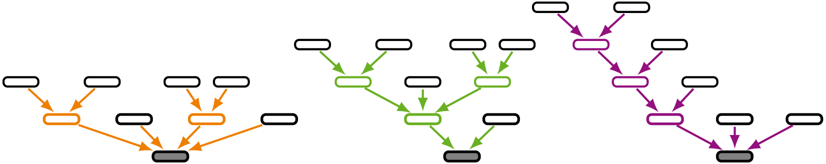
- cutwidth is NP-complete for arbitrary graphs (Bodlaender, Fellows, and Thilikos 2009)
- directed cutwidth on trees can be computed in polynomial time (Theorem 1)

How can we display and measure proofs?



- bushiness score: $\frac{\text{size}}{\text{depth}+1}$ ("average number of nodes per level")
- average step complexity (Horridge, Bail, Parsia, and Sattler 2013)

What does this mean for our example?



	TEXTBOOK	ELK	ENVELOPE	
size	**9**	10	10	
depth	**2**	3	4	
cutwidth	4	**3**	**3**	
bushiness	3	2.5	**2**	
complexity	230	**215**	257.5	/ 410

How can we generate the proofs?

$$\left\{ \begin{array}{cc} \text{hasPaperAt} \sqsubseteq \text{contributesTo} & \text{DL25Author} \sqsubseteq \exists\text{hasWorkshopPaperAt.DL25} \\ \exists\text{contributesTo.DL} \sqsubseteq \text{DLCommunityMember} & \text{DL25FullPaperAuthor} \sqsubseteq \text{DL25Author} \\ \text{hasWorkshopPaperAt} \sqsubseteq \text{hasPaperAt} & \text{DL25} \sqsubseteq \text{DL} \end{array} \right\}$$

$$\Downarrow$$

RDF triples

```
:hasPaperAt rdf:type owl:ObjectProperty .
:hasPaperAt rdfs:subPropertyOf :contributesTo .

:DL25Author rdf:type owl:Class .
:DL25Author rdfs:subClassOf _:1 .
_:1 rdf:type owl:Restriction .
_:1 owl:onProperty :hasWorkshopPaperAt .
_:1 owl:someValuesFrom :DL25 .

...
```

How can we generate the proofs?

$$\mathsf{R}_{\sqsubseteq}\ \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$$

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$$\Downarrow$$

Existential rules

```
...
inf:subClassOf(?C,?E) :- inf:subClassOf(?C,?D), nf:subClassOf(?D,?E) .
inf:subClassOf(?E,?Y) :- inf:ex(?E,?R,?C), inf:subClassOf(?C,?D), nf:subProp(?R,?S),
                          nf:exists(?Y,?S,?D), nf:isSubClass(?Y) .
...
```

How can we generate the proofs?

RDF triples
Existential rules

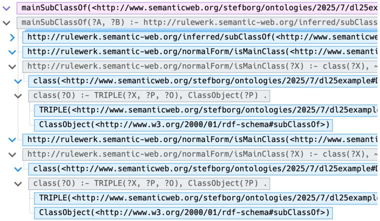
⇒

Nemo
Graph Rule Engine

(Ivliev, Gerlach, Meusel, Steinberg, and Krötzsch 2024)

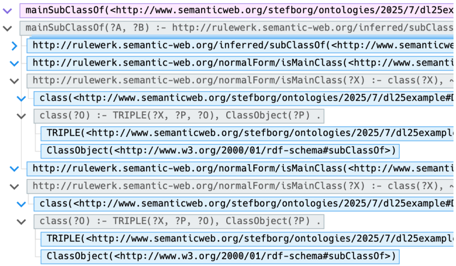
⇒

Trace



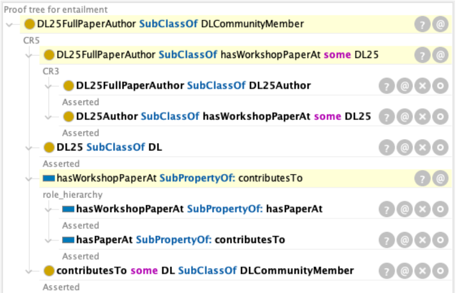
How can we generate the proofs?

Trace



$\Rightarrow$

DL Proof



Which proofs are better?

1573 prototypical $\mathcal{ELH}_\bot$ reasoning tasks from OWL Reasoner Evaluation 2015

(Parsia, Matentzoglu, Gonçalves, Glimm, and Steigmiller 2017)

justifications of average size 10.7 (max. 25), average complexity 495 (max. 1470)

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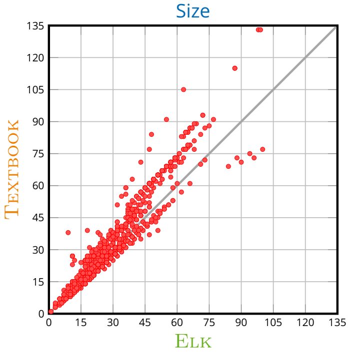

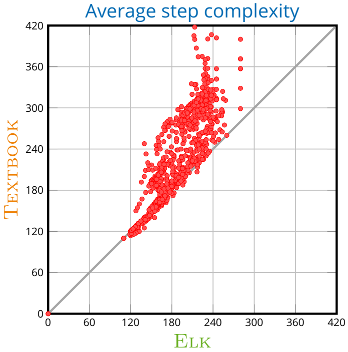

Elk proofs are smaller and easier to understand

Which proofs are better?

1573 prototypical $\mathcal{ELH}_\perp$ reasoning tasks from OWL Reasoner Evaluation 2015

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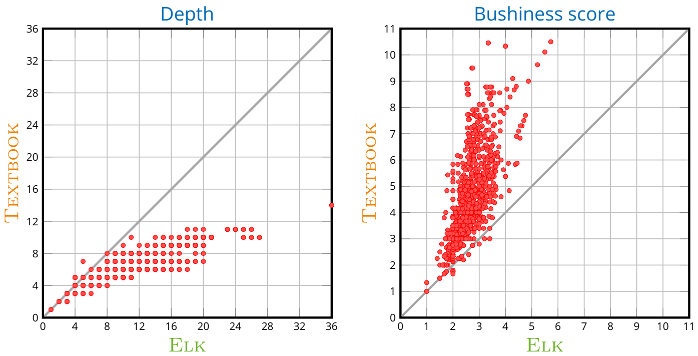
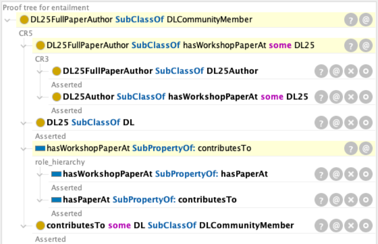


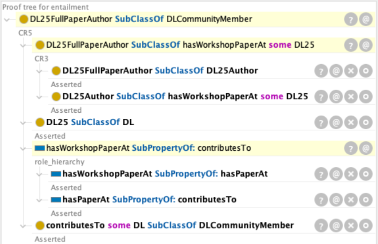
Textbook proofs are better for tree-like views

How can you use this right now?



`github.com/de-tu-dresden-inf-lat/evee`

How can you use this right now?



`github.com/de-tu-dresden-inf-lat/evee`

`github.com/knowsys/nemo`

Nemo
Graph Rule Engine

What comes next?

- Generate all traces with NEMO, not just one
- Compare proofs from ELK calculus to proofs from ELK reasoner

  (Kazakov and Klinov 2014; Kazakov, Klinov, and Stupnikov 2017)
- Calculi for more expressive (Horn and non-Horn) logics

Thank you!

References I

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Kazakov, Yevgeny, Pavel Klinov, and Alexander Stupnikov (2017). "Towards Reusable Explanation Services in Protege". In: *Proceedings of the 30th International Workshop on Description Logics (DL'17)*. URL: http://ceur-ws.org/Vol-1879/paper31.pdf.

Kazakov, Yevgeny, Markus Krötzsch, and František Simančík (2014). "The Incredible ELK: From Polynomial Procedures to Efficient Reasoning with $\mathcal{EL}$ Ontologies". In: *J. Autom. Reason.* 53.1. DOI: 10.1007/s10817-013-9296-3.

References II

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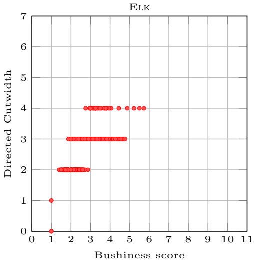
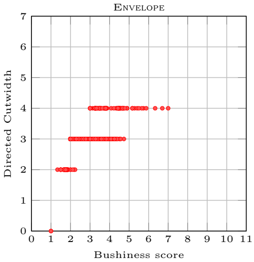
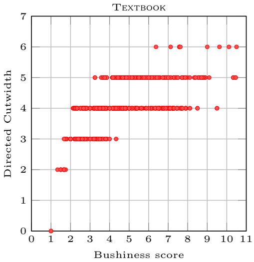
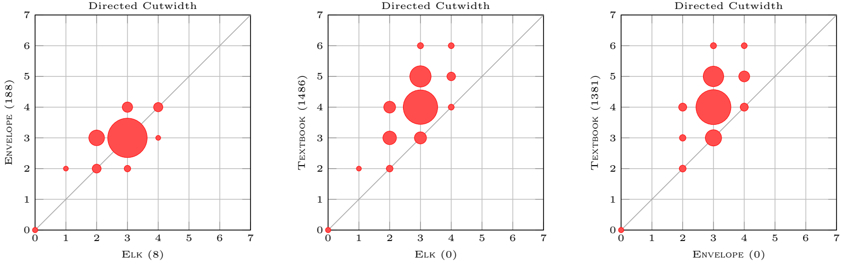
Relation between cutwidth and bushiness



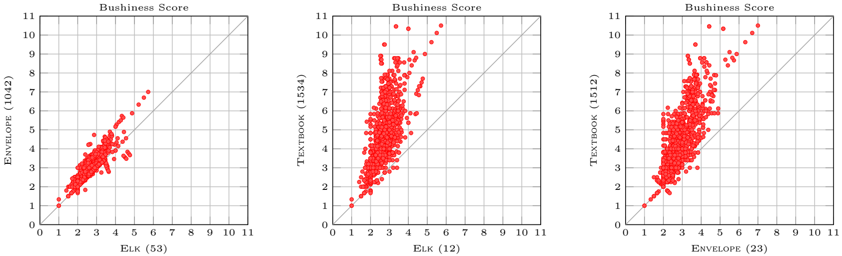




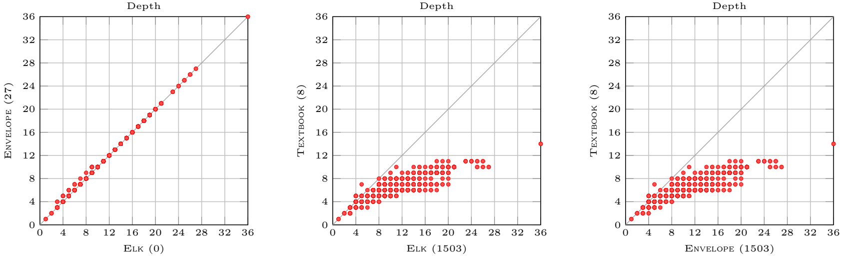
Directed cutwidth



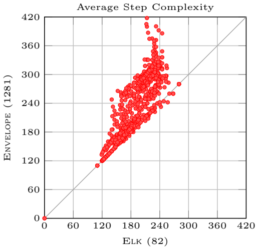
Bushiness score



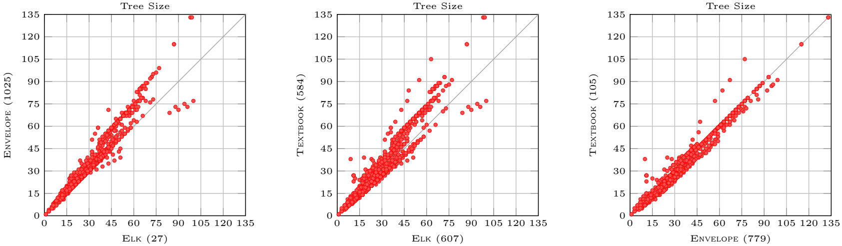
Depth


Depth
Envelope (27)
Elk (0)
Depth
Textbook (8)
Elk (1503)
Depth
Textbook (8)
Envelope (1503)
0
4
8
12
16
20
24
28
32
36

Average step complexity



Tree size


Tree Size
Envelope (1025)
Elk (27)
Tree Size
Textbook (584)
Elk (607)
Tree Size
Textbook (105)
Envelope (779)
0
15
30
45
60
75
90
105
120
135