Dresden University of Technology

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Finding New Boxes for the Diamonds: On the Behavior of Convex Modal Operators for Temporal (Description) Logics

DL'26, Lisboa, Portugal, 19th July, 2026

Dealing with incomplete, noisy data

$\Diamond$Error $\sqsubseteq$ Alarm

"a past error raises an alarm"

Dealing with incomplete, noisy data

$\diamondminus \text{Error} \sqsubseteq \text{Alarm}$

“a past error raises an alarm”

$\diamondminus_2 \text{Error} \sqsubseteq \text{Alarm}$

“an intermittent error raises an alarm”

Dealing with incomplete, noisy data

$\Diamond\mathrm{Error} \sqsubseteq \mathrm{Alarm}$

“a past error raises an alarm”

$\Diamond_2\mathrm{Error} \sqsubseteq \mathrm{Alarm}$

“an intermittent error raises an alarm”

$\Box_2\mathrm{Error} \sqsubseteq \mathrm{Alarm}$

“a non-spurious error raises an alarm”

Convex diamond operators

Closure operators $2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$:

(Borgwardt, Forkel, and Kovtunova 2019)

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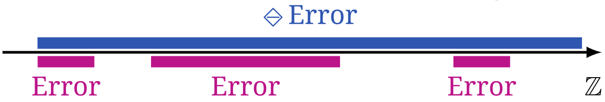

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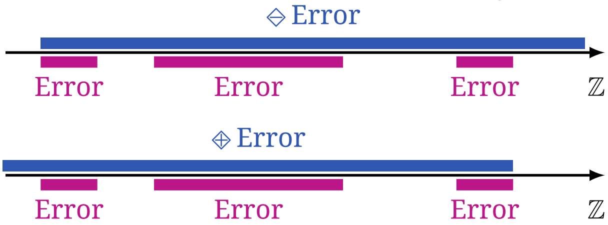

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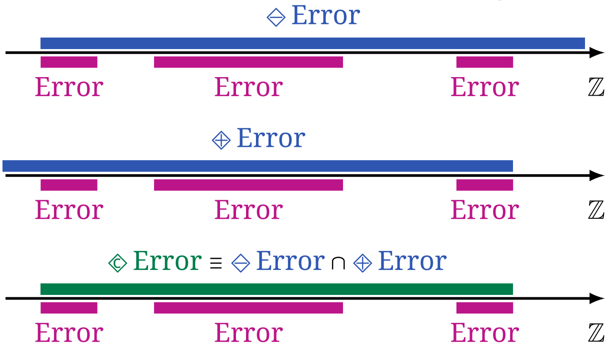

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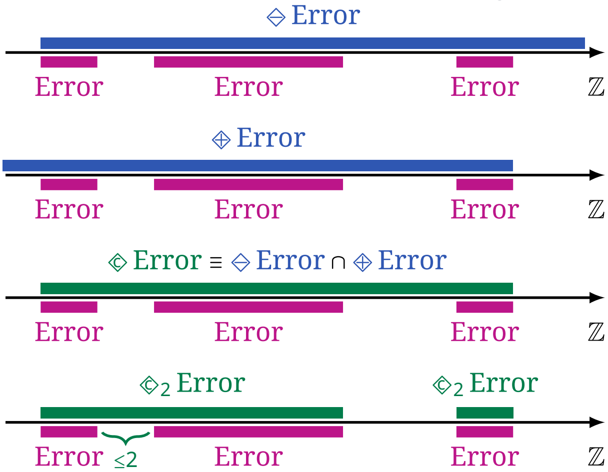

A temporal description logic with convex diamonds

$\mathcal{TELH}_{\bot}^{\circledcirc,\mathsf{lhs}}$: (Gutiérrez-Basulto, Jung, and Kontchakov 2016; Borgwardt, Forkel, and Kovtunova 2019)

$\Diamond A \sqsubseteq B$ $\quad A_1 \sqcap A_2 \sqsubseteq B$ $\quad \Diamond r \sqsubseteq s$ $\quad \Diamond A \sqsubseteq \exists r.B$ $\quad \exists r.A \sqsubseteq B$ $\quad A(a,i)$ $\quad r(a,b,i)$

$\Diamond \in \{\circledcirc_0, \circledcirc_1, \circledcirc_2, \dots, \circledcirc, \oplus, \ominus, \pm\}$ $\quad i \in \mathbb{Z}$

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$$\Diamond \in \{\circledcirc_0, \circledcirc_1, \circledcirc_2, \dots, \circledcirc, \Diamond\!\!\!+, \ominus, \pm\} \qquad i \in \mathbb{Z}$$

Reasoning is P-complete:

$$\frac{\Diamond A_1 \sqsubseteq A_2 \qquad \Diamond\!\!\cdot A_2 \sqsubseteq A_3}{(\Diamond\!\!\cdot \circ \Diamond)A_1 \sqsubseteq A_3} \qquad \frac{\Diamond A \sqsubseteq A_1 \qquad \Diamond\!\!\cdot A \sqsubseteq A_2 \qquad A_1 \sqcap A_2 \sqsubseteq B}{(\Diamond \cap \Diamond\!\!\cdot)A \sqsubseteq B} \qquad \dots$$

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- composition $\circ$, component-wise intersection $\cap$, union $\cup$
- $\{\circledC_0, \circledC_1, \circledC_2, \dots, \circledC, \oplus, \ominus, \pm\}$ is closed under $\circ, \cap, \cup$

A temporal description logic with convex diamonds

$\mathcal{TELH}_{\bot}^{\Diamond^{c},\text{lhs}}$: (Gutiérrez-Basulto, Jung, and Kontchakov 2016; Borgwardt, Forkel, and Kovtunova 2019)

$$\Diamond A \sqsubseteq B \qquad A_1 \sqcap A_2 \sqsubseteq B \qquad \Diamond r \sqsubseteq s \qquad \Diamond A \sqsubseteq \exists r.B \qquad \exists r.A \sqsubseteq B \qquad A(a,i) \qquad r(a,b,i)$$

$$\Diamond \in \{\Diamond^{c}_0, \Diamond^{c}_1, \Diamond^{c}_2, \dots, \Diamond^{c}, \Diamond^{+}, \Diamond^{-}, \Diamond^{\pm}\} \qquad i \in \mathbb{Z}$$

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- $\Diamond A \sqsubseteq B$ entails $\Diamond^{\cdot} A \sqsubseteq B$ iff $\Diamond^{\cdot} \subseteq \Diamond$ ($\Diamond^{\cdot} M \subseteq \Diamond M$ for all $M \subseteq \mathbb{Z}$)

A temporal description logic with convex diamonds

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- $\Diamond A \sqsubseteq B$ entails $\Diamond^{\cdot} A \sqsubseteq B$ iff $\Diamond^{\cdot} \subseteq \Diamond$ $\quad$ ($\Diamond^{\cdot} M \subseteq \Diamond M$ for all $M \subseteq \mathbb{Z}$)

$$\Diamond^{c}_0 \subset \Diamond^{c}_1 \subset \cdots \subset \Diamond^{c}_n \subset \Diamond^{c}_{n+1} \subset \cdots \subset \Diamond^{c} \subset \Diamond^{-} \subset \Diamond^{\pm}, \qquad \Diamond^{c} \subset \Diamond^{+} \subset \Diamond^{\pm}$$

Convex box operators

$\boxed{\mathrm{c}}_n \mathrm{Error} \equiv \neg \diamond\!\!\!\!\!\raisebox{0.1ex}{\tiny c}\;_n \neg \mathrm{Error}$

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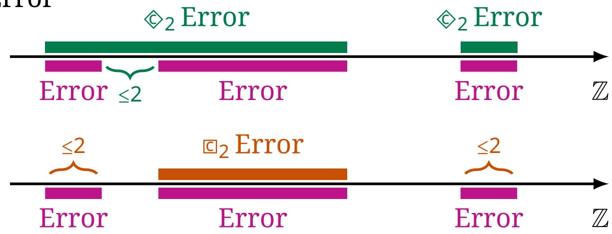

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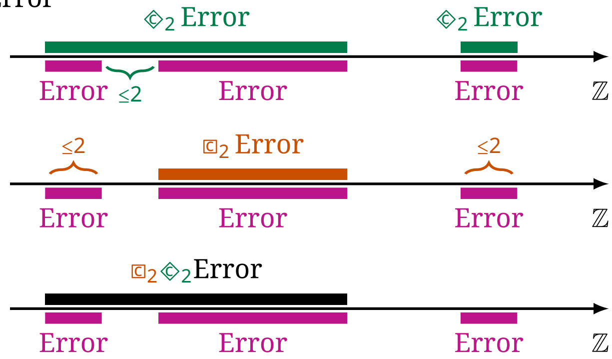
$\boxed{\text{c}}_n\,\mathrm{Error} \equiv \neg \diamondsuit_n \neg \mathrm{Error}$



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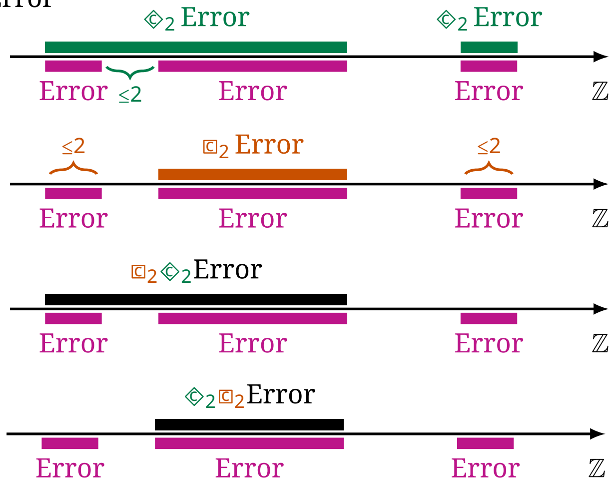
$\boxed{c}_n \mathrm{Error} \equiv \neg \diamondsuit^{c}_n \neg \mathrm{Error}$



Convex box operators

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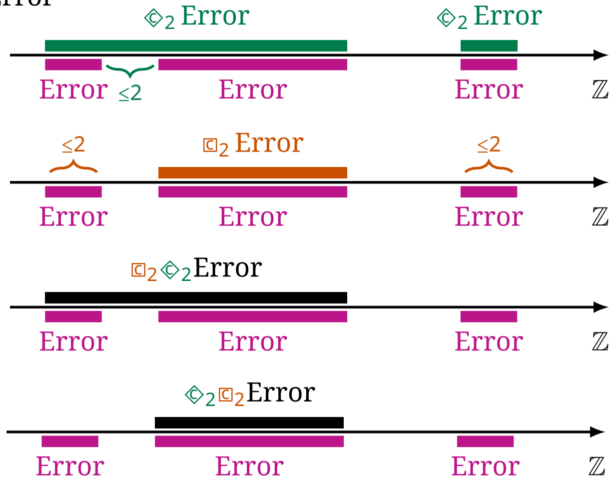

$\Diamond_2 \Box_2 \subseteq \Box_2 \Diamond_2$

Combined temporal operators

Before we can tackle $\mathcal{TELH}_{\bot}^{\diamondsuit\!\!\!\!\bigcirc,\boxdot,\mathrm{lhs}}$,
we need to characterize $\subseteq$ between combinations ($\circ, \cap, \cup$) of convex operators $\diamondsuit\!\!\!\!\bigcirc_n, \boxdot_n$.

Combined temporal operators

Before we can tackle $\mathcal{TELH}_{\bot}^{\Diamond_{\mathsf{c}},\boxed{\mathsf{c}},\mathsf{lhs}}$,
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$$\begin{tikzcd}[column sep=small, row sep=small]
 & \boxminus \arrow[dr, phantom, "\subset" sloped] & & & & & & & & & & & & & & & & & & & & \Diamond_{\ominus} \arrow[dr, phantom, "\subset" sloped] & \\
\boxed{\pm} \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \boxed{\mathsf{c}} \subset \cdots \subset \boxed{\mathsf{c}}_{n+1} \subset \boxed{\mathsf{c}}_n \subset \cdots \subset \boxed{\mathsf{c}}_1 \subset \boxed{\mathsf{c}}_0 = \Diamond_{\mathsf{c}\,0} \subset \Diamond_{\mathsf{c}\,1} \subset \cdots \subset \Diamond_{\mathsf{c}\,n} \subset \Diamond_{\mathsf{c}\,n+1} \subset \cdots \subset \Diamond_{\mathsf{c}} \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & & & & & & & & & & & & & & & & & & & \Diamond_{\pm} \\
 & \boxplus \arrow[ur, phantom, "\subset" sloped] & & & & & & & & & & & & & & & & & & & & \Diamond_{\oplus} \arrow[ur, phantom, "\subset" sloped] &
\end{tikzcd}$$

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Before we can tackle $\mathcal{TELH}_\bot^{\Diamond^{\mathsf{c}},\Box^{\mathsf{c}},\mathrm{lhs}}$,
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$$
\begin{tikzcd}[column sep=small, row sep=small]
 & \boxminus \arrow[dr, phantom, "\subset" sloped] & & & & & & & & & & & & & & & & & & & \Diamond^{-} \arrow[dr, phantom, "\subset" sloped] & \\
\boxed{\pm} \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \Box^{\mathsf{c}} \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & \Box^{\mathsf{c}}_{n+1} \arrow[r, phantom, "\subset"] & \Box^{\mathsf{c}}_n \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & \Box^{\mathsf{c}}_1 \arrow[r, phantom, "\subset"] & \Box^{\mathsf{c}}_0 \arrow[r, phantom, "="] & \Diamond^{\mathsf{c}}_0 \arrow[r, phantom, "\subset"] & \Diamond^{\mathsf{c}}_1 \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & \Diamond^{\mathsf{c}}_n \arrow[r, phantom, "\subset"] & \Diamond^{\mathsf{c}}_{n+1} \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & \Diamond^{\mathsf{c}} \arrow[urrrrr, phantom, "\subset" sloped] \arrow[drrrrr, phantom, "\subset" sloped] & & & & & & \Diamond^{\pm} \\
 & \boxplus \arrow[ur, phantom, "\subset" sloped] & & & & & & & & & & & & & & & & & & & \Diamond^{+} \arrow[ur, phantom, "\subset" sloped] &
\end{tikzcd}
$$

Lemma (Normal form)

Every composition sequence of these temporal operators is equivalent to one of the form

$$\Diamond^{\odot}\,\Box_k\Diamond_k\ldots\Box_1[\Diamond_1]$$

with alternating convex parts $\Box_k \subset \cdots \subset \Box_1$ *and* $\Diamond_k \supset \cdots \supset \Diamond_1$ *and classical part* $\Diamond^{\odot} \in \{\epsilon, \Diamond^{-}, \Diamond^{+}, \Diamond^{\pm}, \boxplus\Diamond^{+}, \boxminus\Diamond^{-}\}$ *or, dually,*

$$\boxdot\,\Diamond_k\Box_k\ldots\Diamond_1[\Box_1].$$

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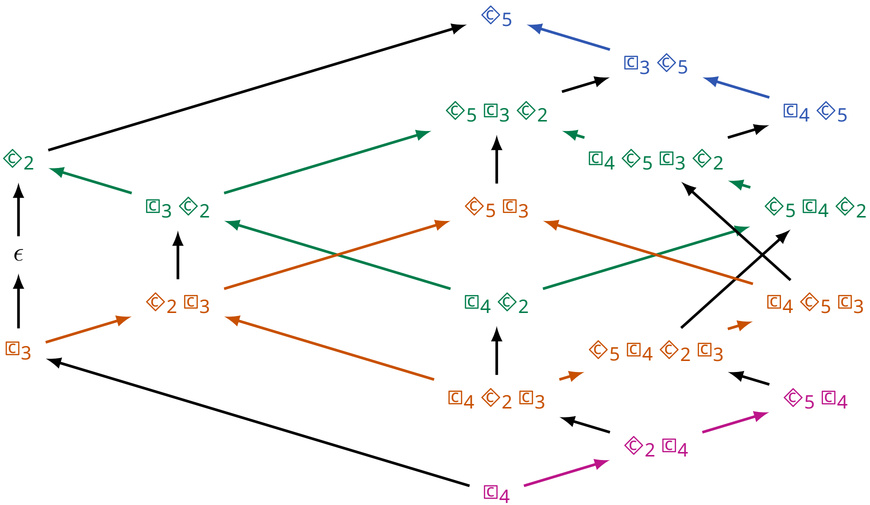
$$\boxdot \Diamond_k \Box_k \ldots \Diamond_1 [\Box_1] .$$

$\Diamond^{c}_{n}\Diamond^{c}_{m} \equiv \Diamond^{c}_{\max(n,m)}$ $\qquad$ $\Box^{c}_{n}\Box^{c}_{m} \equiv \Box^{c}_{\max(n,m)}$ $\qquad$ $\Box^{c}_{k}\Diamond^{c}_{\ell}\Box^{c}_{m} \equiv \Diamond^{c}_{\ell}\Box^{c}_{m}$ if $k \le m$

Subsumption between sequences



Finite-state transducer for sequences

$$\boxed{C}4 \quad \Diamond_{C}5 \quad \boxed{C}3 \quad \Diamond_{C}2$$

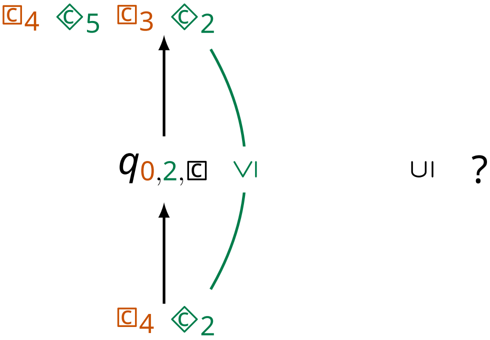
$$\uparrow$$

$$q_0 \qquad\qquad \subseteq \quad ?$$

$$\uparrow$$

$$\boxed{C}4 \quad \Diamond_{C}2$$

Finite-state transducer for sequences



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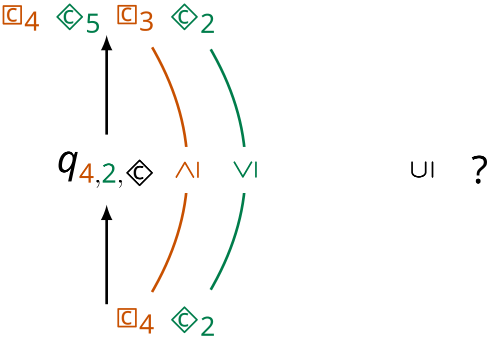

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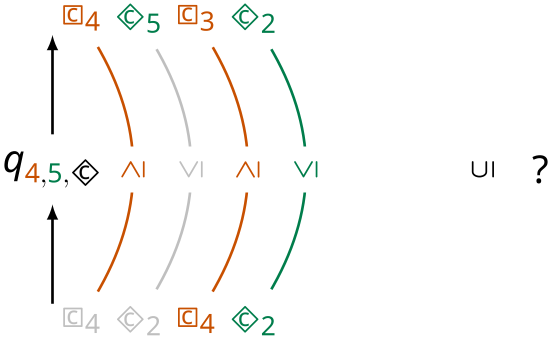

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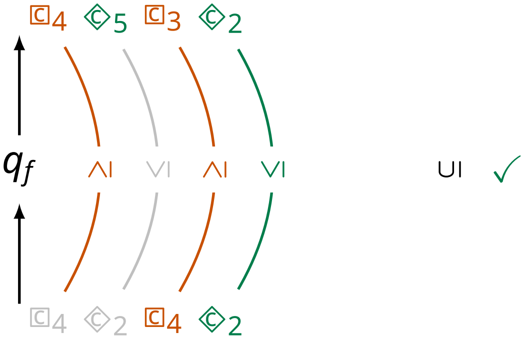
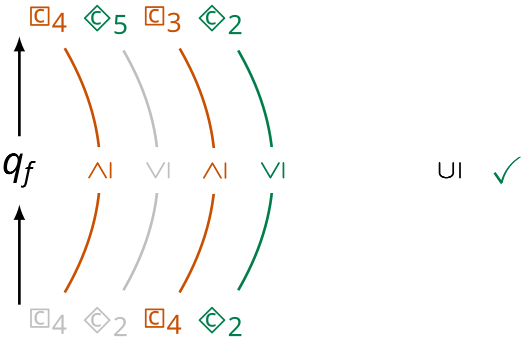


Finite-state transducer for sequences



Lemma (Subsumption between sequences)

Subsumption between composition sequences of convex operators can be decided in polynomial time.

Subsumption between complex formulas

$\Diamond_0 \cup \Box_2 \Diamond_1 \equiv \Diamond_1 \qquad \Box_3 \equiv \Box_5 \cap \Diamond_7 \Box_3$

Subsumption between complex formulas

$$\Diamond_0 \cup \Box_2 \Diamond_1 \equiv \Diamond_1 \qquad \Box_3 \equiv \Box_5 \cap \Diamond_7 \Box_3$$

$$\begin{aligned}
\mathsf{reach}(\Box_n) := \mathsf{reach}(\Diamond_n) &:= n \\
\mathsf{reach}(\phi \circ \psi) &:= \mathsf{reach}(\phi) + \mathsf{reach}(\psi) \\
\mathsf{reach}(\phi \cup \psi) &:= \max\{\mathsf{reach}(\phi), \mathsf{reach}(\psi)\} \\
\mathsf{reach}(\phi \cap \psi) &:= \max\{\mathsf{reach}(\phi), \mathsf{reach}(\psi)\}
\end{aligned}$$

Subsumption between complex formulas

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\end{aligned}$$

Lemma

For a $(\circ, \cap, \cup)$*-formula* ϕ *over convex operators (excluding* $\Box$, $\Diamond$*) with* $\mathrm{reach}(\phi) = n$*, we have* $i \in \phi M$ *iff* $i \in \phi(M \cap [i-n, i+n])$ *for every* $M \subseteq \mathbb{Z}$ *and* $i \in M$.

Subsumption between complex formulas

$$\diamondsuit_0 \cup \boxdot_2 \diamondsuit_1 \equiv \diamondsuit_1 \qquad \boxdot_3 \equiv \boxdot_5 \cap \diamondsuit_7 \boxdot_3$$

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\text{reach}(\phi \circ \psi) &:= \text{reach}(\phi) + \text{reach}(\psi) \\
\text{reach}(\phi \cup \psi) &:= \max\{\text{reach}(\phi), \text{reach}(\psi)\} \\
\text{reach}(\phi \cap \psi) &:= \max\{\text{reach}(\phi), \text{reach}(\psi)\}
\end{aligned}$$

Lemma

For a $(\circ, \cap, \cup)$-formula ϕ over convex operators (excluding $\boxdot$, $\diamondsuit$) with reach$(\phi) = n$, we have $i \in \phi M$ iff $i \in \phi(M \cap [i-n, i+n])$ for every $M \subseteq \mathbb{Z}$ and $i \in M$.

Lemma

Subsumption between $(\circ, \cap, \cup)$-formulas over convex operators (excluding $\boxdot$, $\diamondsuit$) can be checked in (double) exponential time, under unary (binary) encoding of numbers.

Summary

- New convex box operators for filtering noisy input data
- Algorithms for subsumption between combinations of convex operators
- P/2ExpTime upper bounds for sequences/formulas

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Future work:

- Extend to classical operators $\diamondplus$, $\boxminus$, ...
- Investigate new temporal description logics with convex box operators

Thank you!

References I

Borgwardt, Stefan, Walter Forkel, and Alisa Kovtunova (2019). "Finding New Diamonds: Temporal Minimal-World Query Answering over Sparse ABoxes". In: *Rules and Reasoning - Third International Joint Conference, RuleML+RR 2019, Proceedings*. DOI: 10.1007/978-3-030-31095-0˙1.

Gutiérrez-Basulto, Víctor, Jean Christoph Jung, and Roman Kontchakov (2016). "Temporalized EL Ontologies for Accessing Temporal Data: Complexity of Atomic Queries". In: *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence (IJCAI)*. URL: http://www.ijcai.org/Abstract/16/160.