TECHNISCHE UNIVERSITÄT DRESDEN
DRESDEN concept
Institut für Theoretische Informatik Lehrstuhl für Automatentheorie

REASONING IN EXPRESSIVE DESCRIPTION LOGICS UNDER INFINITELY VALUED GÖDEL SEMANTICS

Stefan Borgwardt and Rafael Peñaloza

Wrocław, 24.09.2015

Description Logics

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$$\exists \text{respiratoryRate}.\text{LowFreq} \sqcap \exists \text{bodyTemperature}.\text{LowTemp}$$

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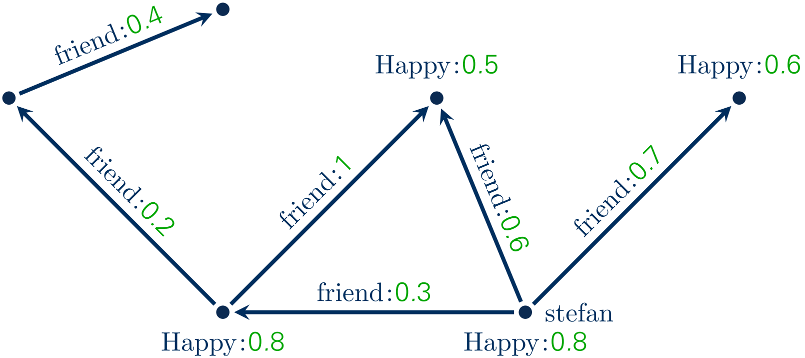
- Gödel t-norm: $\min\{x, y\}$
- Gödel implication: $x \Rightarrow y := \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases}$
- involutive negation: $1 - x$

G-$\mathcal{SROIQ}$

Interpretations:



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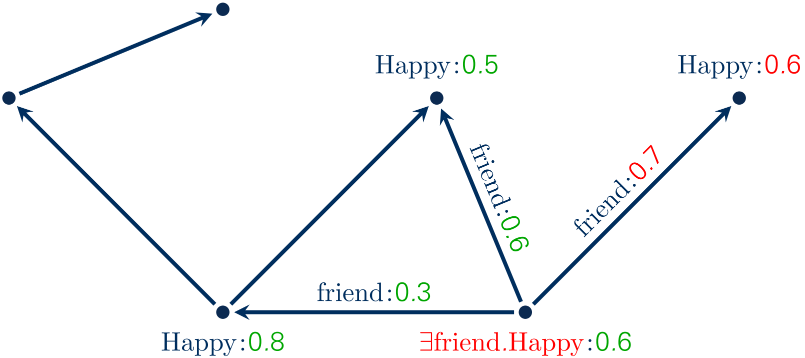

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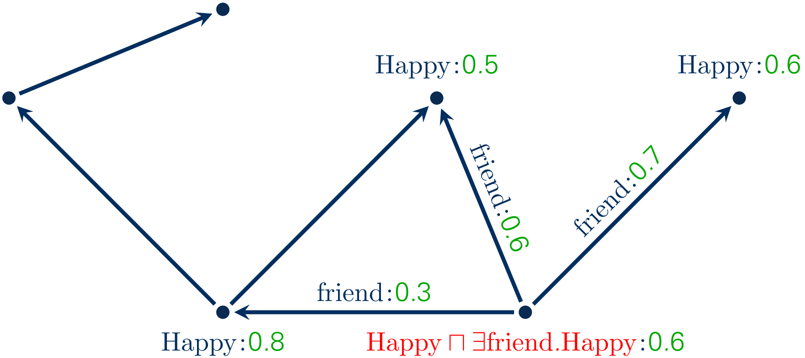

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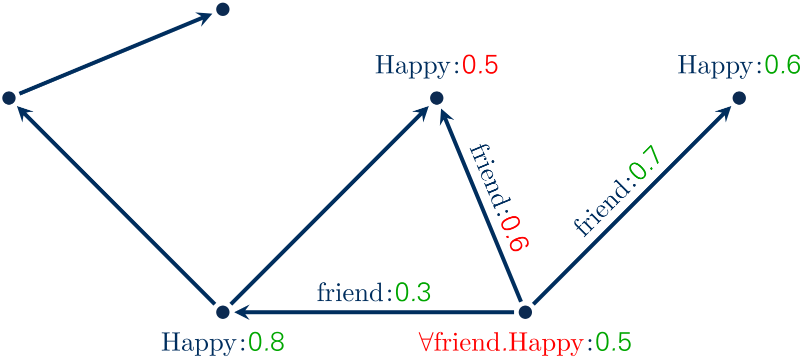

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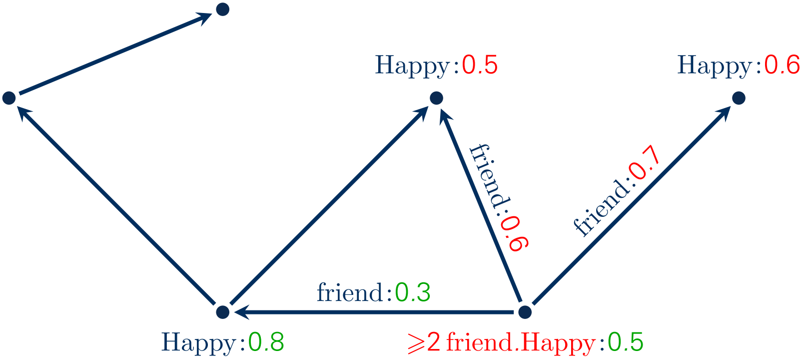

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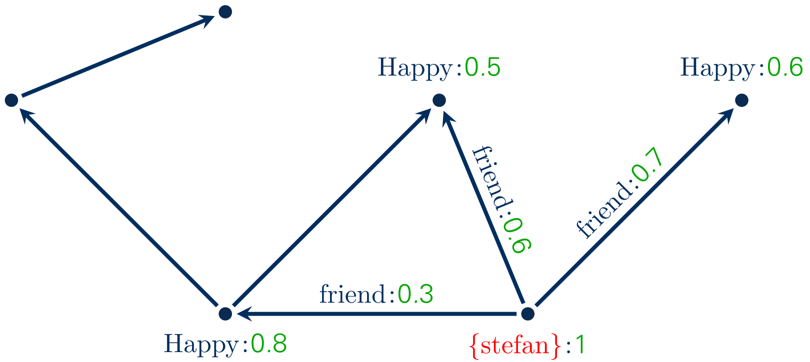

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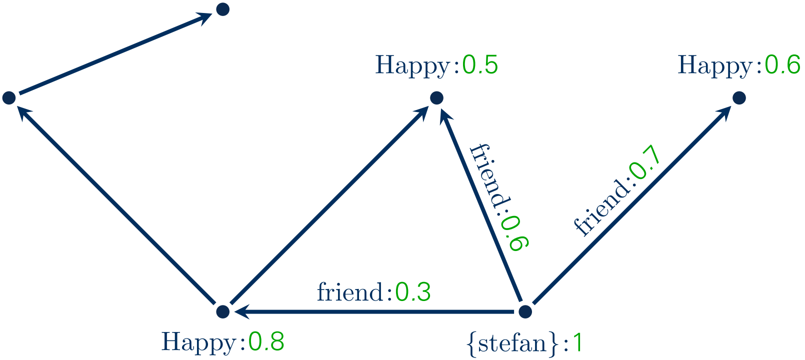

Ontologies: $\langle \text{stefan}:\text{Happy} \geqslant \text{rafael}:\text{Happy} \rangle$

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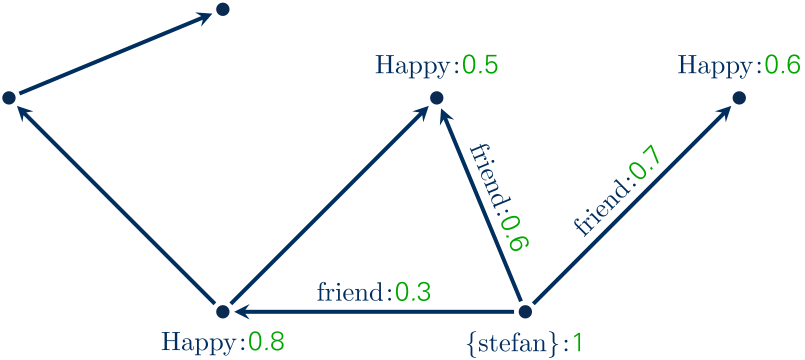

Ontologies: $\langle \text{Happy} \sqsubseteq \neg\text{Sad} \geqslant 0.4 \rangle$

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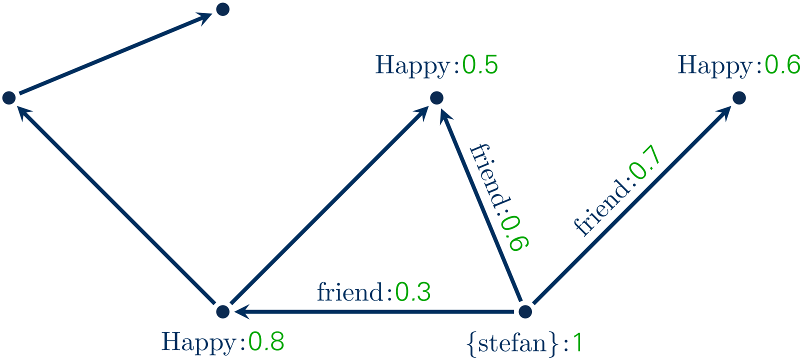

Ontologies: $\langle \text{friend friend} \sqsubseteq \text{friend} \geqslant 0.7 \rangle$

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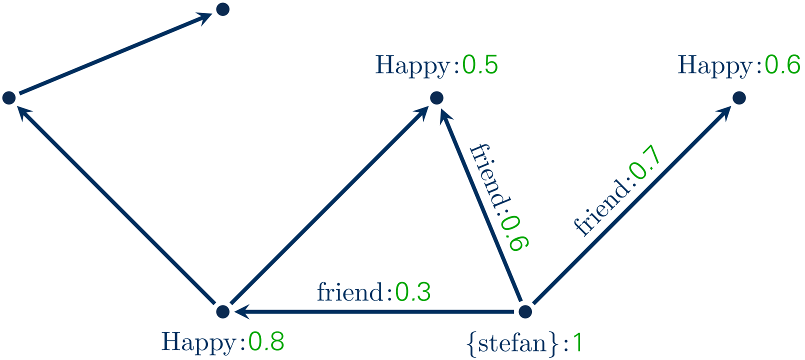

Ontologies: $\langle \text{friend}^- \sqsubseteq \text{friend} \geqslant 0.8 \rangle$

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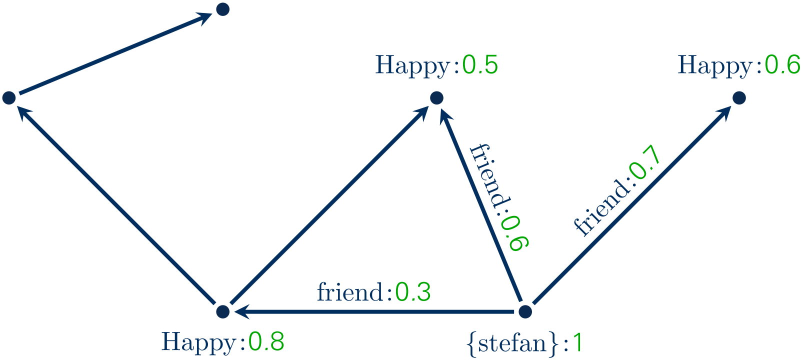

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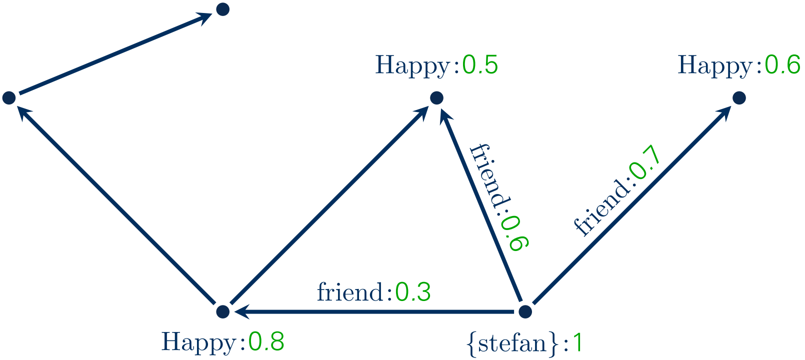

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Consistency: Does the ontology have a model?

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Other Semantics (not Gödel):

- undecidable already in $\mathcal{ALC}$
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Finitely Valued Gödel ($\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$):

- decidable by encoding into classical $\mathcal{SROIQ}$

$$\text{Happy} \leadsto \text{Happy}_{\geqslant \frac{1}{n-1}}, \text{Happy}_{\geqslant \frac{2}{n-1}}, \ldots$$

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Infinitely Valued Gödel ($[0, 1]$):

- trivial if there is no negation and only assertions of the form
  $\langle \mathrm{stefan} : \mathrm{Happy} \geqslant 0.5 \rangle$
- decidable for $\mathcal{ALC}$ using an automata-based algorithm

Two Problems with Models

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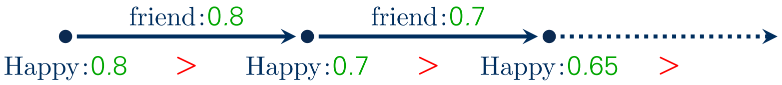

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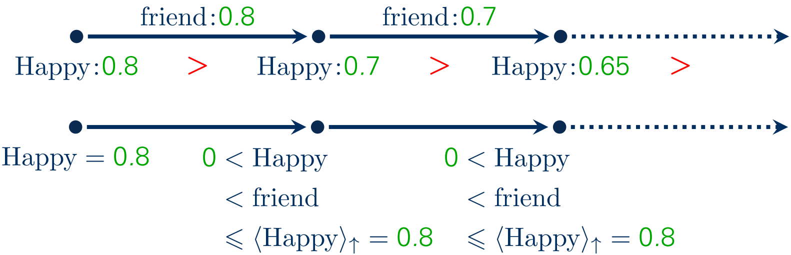

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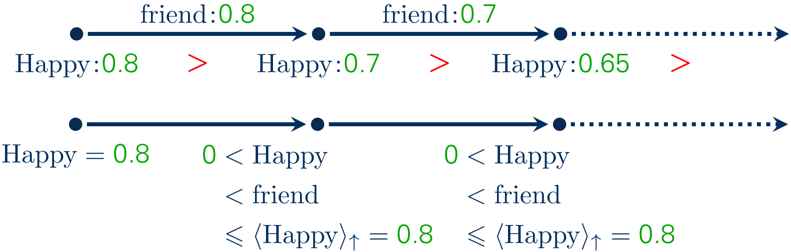

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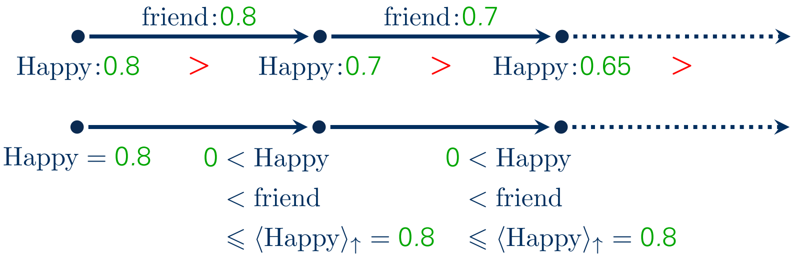
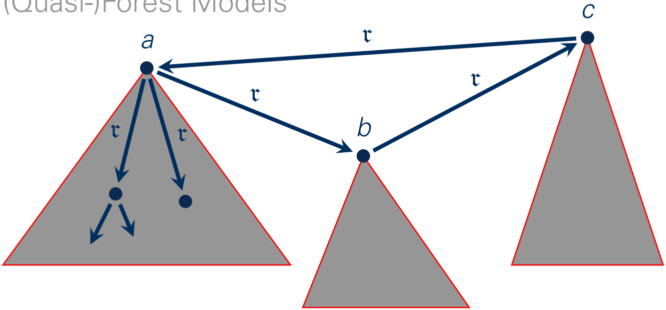


2. G-$\mathcal{SROIQ}$ does not have the forest model property.

Remove nominals ($\mathcal{O}$), inverse roles ($\mathcal{I}$), or number restrictions ($\mathcal{Q}$):

$\rightsquigarrow$ G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, G-$\mathcal{SROI}$

(Quasi-)Forest Models



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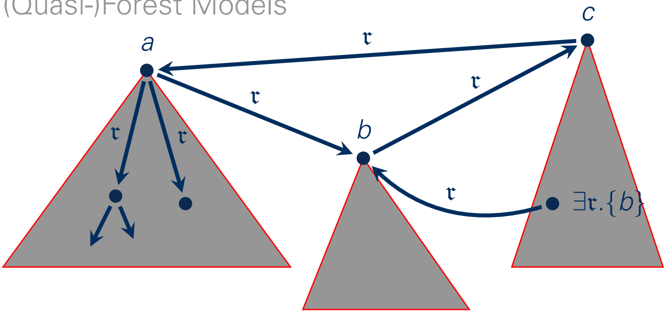

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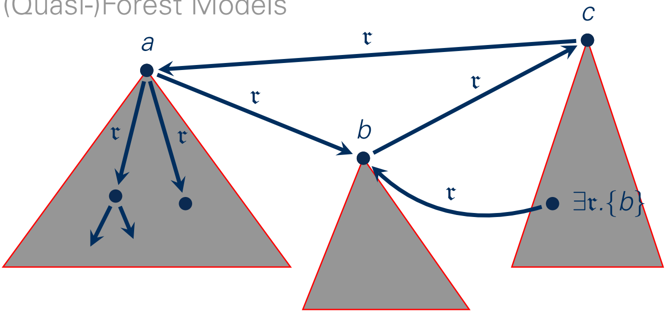

Total preorders over

- all values of the ontology — 0.8
- all subconcepts of the ontology — $\mathrm{Happy}, \forall\mathrm{friend}.\mathrm{Happy}$
- all subconcepts of the parent — $\langle\mathrm{Happy}\rangle_\uparrow, \langle\forall\mathrm{friend}.\mathrm{Happy}\rangle_\uparrow$
- all assertions — $\mathrm{stefan}:\mathrm{Happy}$
- all roles to the parent node — $\mathrm{friend}, \mathrm{friend}^-$
- all roles to named individuals — $(*, \mathrm{stefan}):\mathrm{friend}$

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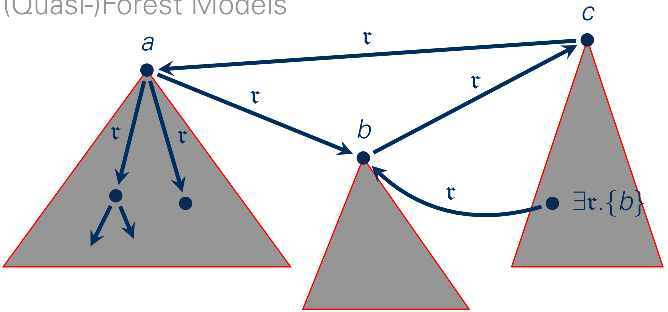

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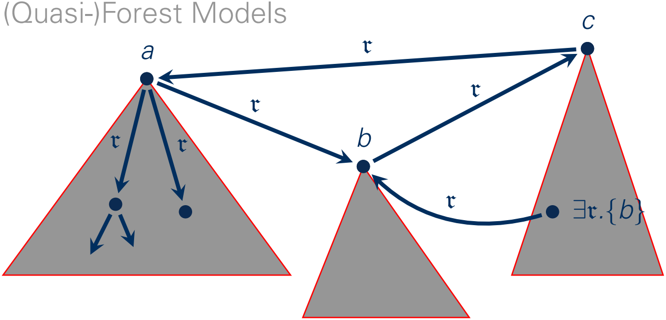

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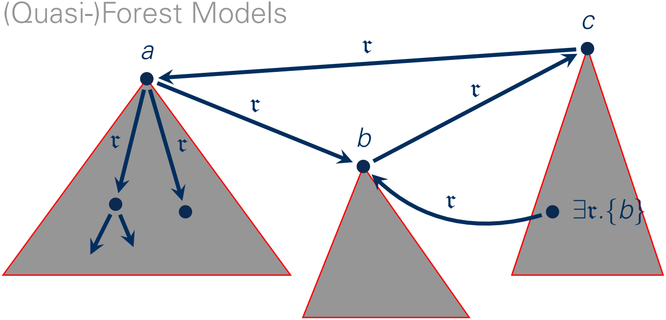

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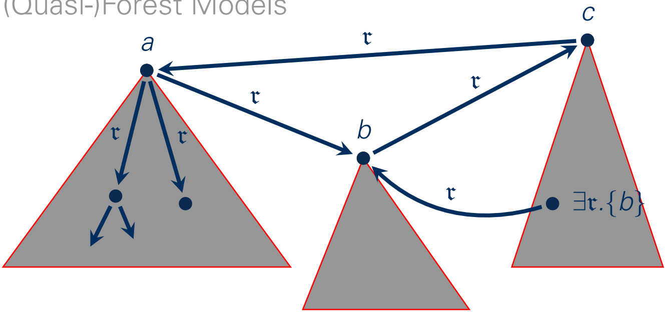

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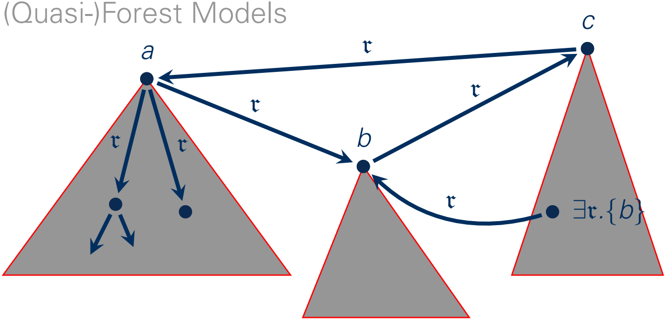

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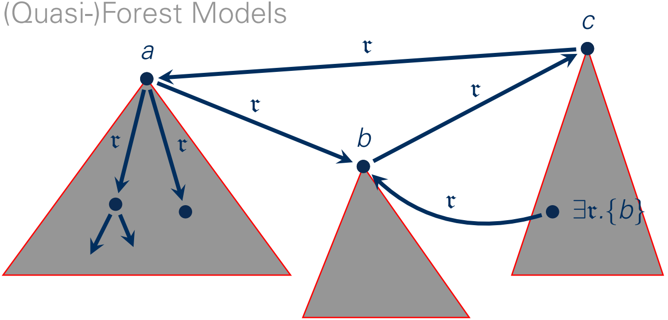

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	classical	Gödel
$\mathcal{ALC}$–$\mathcal{SHOI}$/$\mathcal{SHIQ}$	ExpTime-c.	ExpTime-c.
$\mathcal{SHOQ}$	ExpTime-c.	in 2-ExpTime
$\mathcal{SROI}$/$\mathcal{SROQ}$	in 2-ExpTime	in 2-ExpTime
$\mathcal{SRIQ}$	2-ExpTime-c.	2-ExpTime-c.
$\mathcal{SROIQ}$	2-NExpTime-c.	?

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