TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institute of Theoretical Computer Science Chair of Automata Theory

POSITIVE SUBSUMPTION IN FUZZY $\mathcal{EL}$ WITH GENERAL T-NORMS

Stefan Borgwardt Rafael Peñaloza

北京, August 7, 2013

Vagueness in SNOMED CT

- $\mathcal{EL}$ is used in SNOMED CT

$$\begin{aligned}\mathsf{PerinatalCyanoticAttack} \sqsubseteq\ & \mathsf{CardiovascularDisorder} \sqcap \\ & \exists \mathsf{occurrence}.\mathsf{PerinatalPeriod} \sqcap \\ & \exists \mathsf{manifestation}.\mathsf{Cyanosis}\end{aligned}$$

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- does reasoning stay tractable?

Mathematical fuzzy logic

all statements hold to a degree from $[0, 1]$

t-norm $\otimes\colon [0, 1] \times [0, 1] \to [0, 1]$ generalizes $\wedge\colon \{0, 1\} \times \{0, 1\} \to \{0, 1\}$:

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- unique if $\otimes$ is continuous
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[Hájek 2001, 2005]

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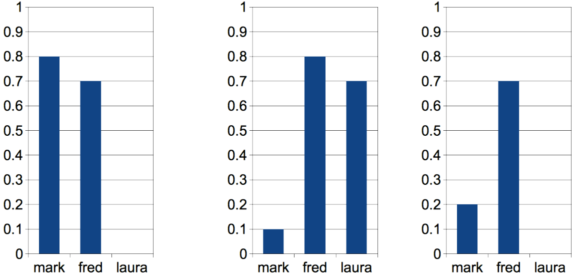

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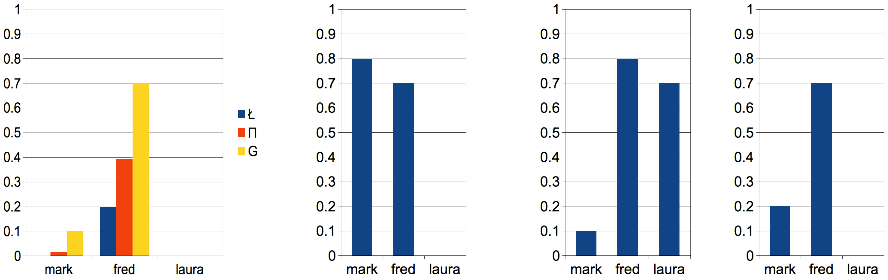

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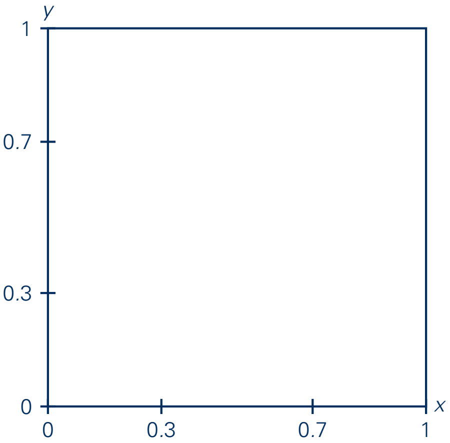

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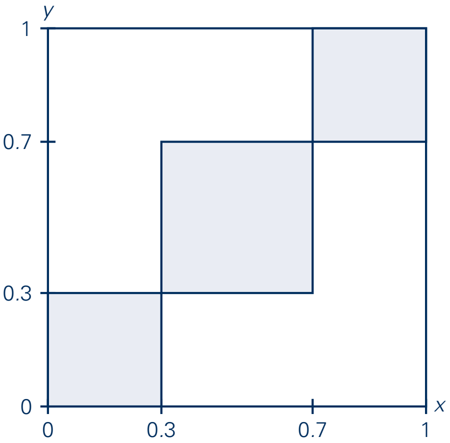

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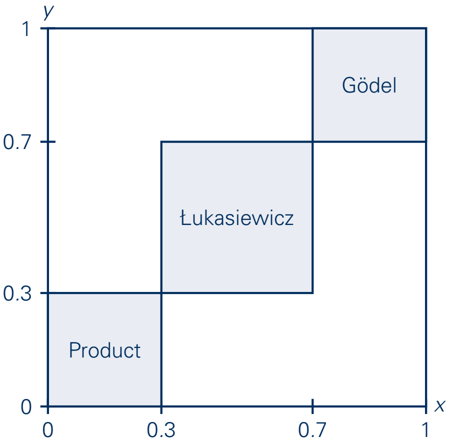

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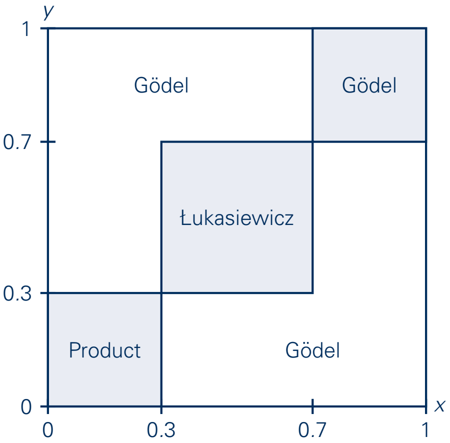

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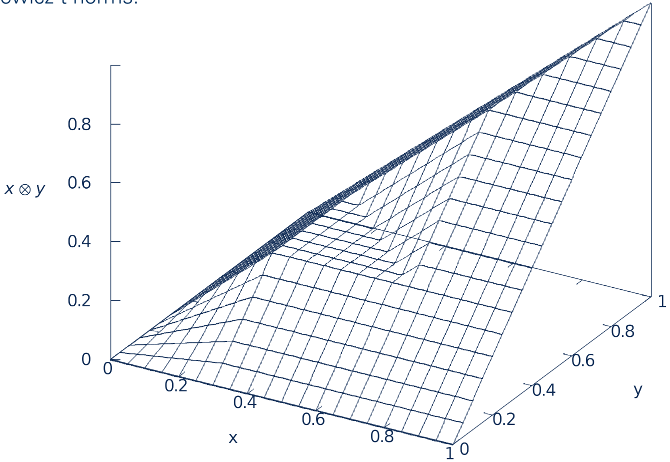

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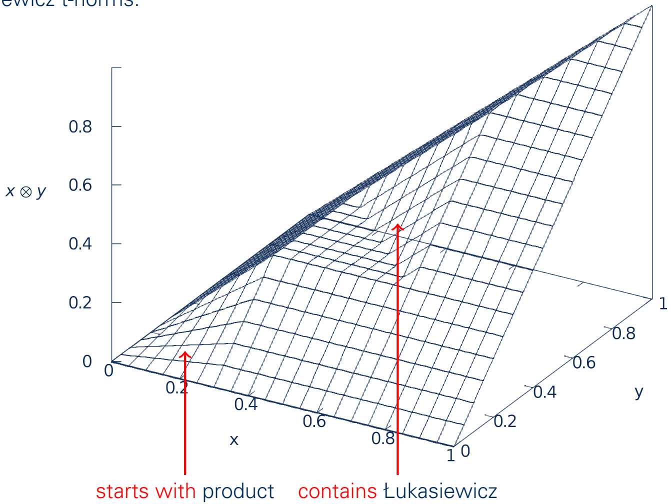
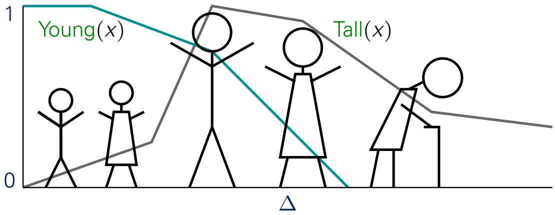


$\otimes$-$\mathcal{EL}$

- concept names: Patient, Disease



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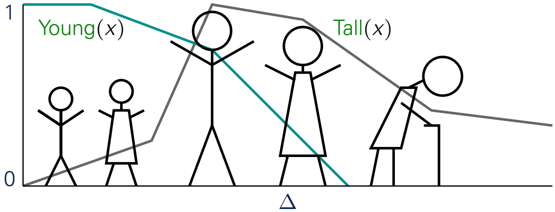

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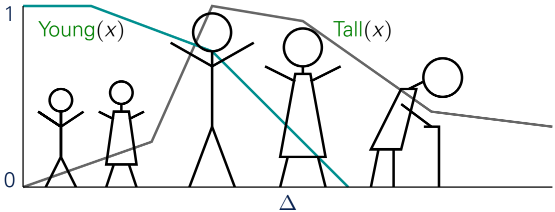

- role names: hasSeverity, hasSymptom
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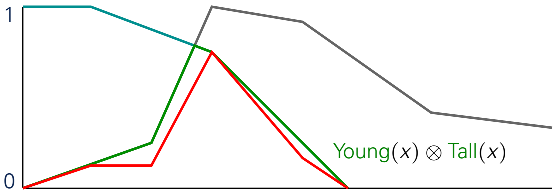
- conjunction: Young $\sqcap$ Tall



$\otimes$-$\mathcal{EL}$ (2)

- existential restriction: $\exists$hasSymptom.Cough

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- TBox $\mathcal{T}$: set of general concept inclusions (GCIs):

$$\langle \text{PerinatalCyanoticAttack} \sqsubseteq \exists\text{hasManifestation.Cyanosis} \geq q\rangle$$

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reasoning tasks:

- *p*-subsumption: does $\langle C \sqsubseteq D \geq p \rangle$ follow from $\mathcal{T}$?
- compute the best subsumption degree (bsd) of C and D!
- classification: compute all bsds between concept names!
- positive subsumption: does “$C(x) \Rightarrow D(x) > 0$” follow from $\mathcal{T}$?

Results

related work:

- classification in classical $\mathcal{EL}$ can be done in polynomial time
  [Baader, Brandt, Lutz 2005]
- several highly optimized reasoners (CEL, jcel, ELK)
- classification in G-$\mathcal{EL}$ can be done in polynomial time
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- 1-subsumption is in P if $\otimes$ does not start with Łukasiewicz, all roles are crisp, and all GCIs are in normal form [DL 2013]

Positive subsumption in Ł-$\mathcal{EL}$ is co-NP-hard

vertex cover problem $\mathcal{V} = (V, \mathcal{E}, k)$:

- $V = \{v_1, \ldots, v_m\}$
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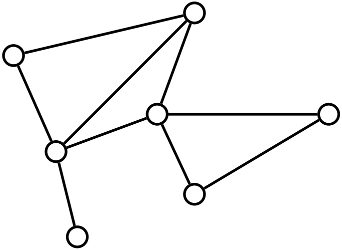

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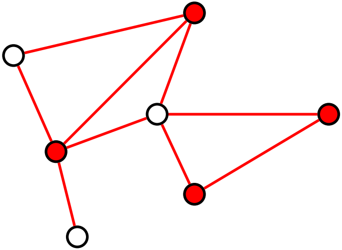

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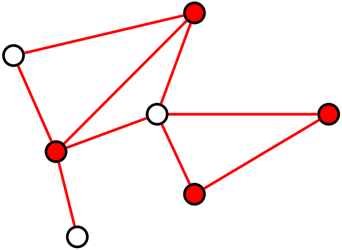

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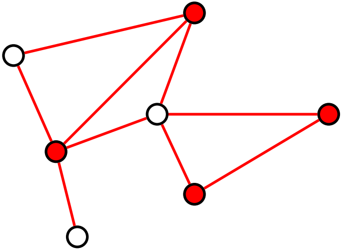

$$\mathcal{T}_{\mathcal{V}} := \{\langle V_i^{m-k} \sqsubseteq V_i^{m-k+1} \geq 1\rangle,\ \langle \top \sqsubseteq V_i \geq \tfrac{m-k-1}{m-k}\rangle \mid 1 \leq i \leq m\} \cup$$

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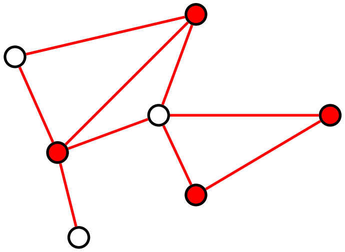

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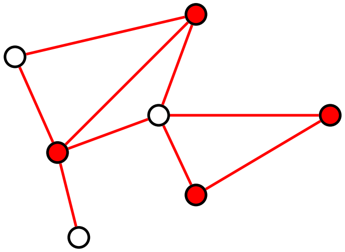

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There is no vertex cover of $(V, \mathcal{E})$ of size $\leq k$ iff "$(V_1 \sqcap \ldots \sqcap V_m)(x) > 0$" (w.r.t. $\mathcal{T}_{\mathcal{V}}$).

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- Π-$\mathcal{EL}$ and Ł-$\mathcal{EL}$ $\rightsquigarrow \langle A \sqcap A \sqsubseteq D \geq o \otimes p \otimes q\rangle$
- need to distinguish A, $A \sqcap A$, A^3, ...

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- classical $\mathcal{EL}$ and G-$\mathcal{EL}$ $\rightsquigarrow \langle A \sqsubseteq D \geq o \otimes p \otimes q\rangle$
- Π-$\mathcal{EL}$ and Ł-$\mathcal{EL}$ $\rightsquigarrow \langle A \sqcap A \sqsubseteq D \geq o \otimes p \otimes q\rangle$
- need to distinguish A, $A \sqcap A$, A^3, ...

polynomial algorithm for 1-subsumption with only crisp roles [DL 2013]

Conclusions

Summary:

- fuzzy variants of the light-weight DL $\mathcal{EL}$
- positive subsumption either co-NP-hard or in P
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Future work:

- completion algorithm for p-subsumption
- upper bounds for co-NP-hard cases

Thank You

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