TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institute of Theoretical Computer Science Chair of Automata Theory

HOW FUZZY IS MY FUZZY DESCRIPTION LOGIC?

Stefan Borgwardt Felix Distel Rafael Peñaloza

Manchester, June 28, 2012

Overview

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Mathematical fuzzy logic

t-norm $\otimes\colon [0,1] \times [0,1] \to [0,1]$ generalizes $\wedge\colon \{0,1\} \times \{0,1\} \to \{0,1\}$:

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[Hájek 2001, 2005]

$\otimes$-$\mathcal{ALC}$ and $\otimes$-$\mathcal{SHOI}$

name	syntax	semantics
concept name	$A \in \mathsf{N_C}$	$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$
role name	$r \in \mathsf{N_R}$	$r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$
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negation	$\neg C$	$\ominus C^\mathcal{I}(x)$
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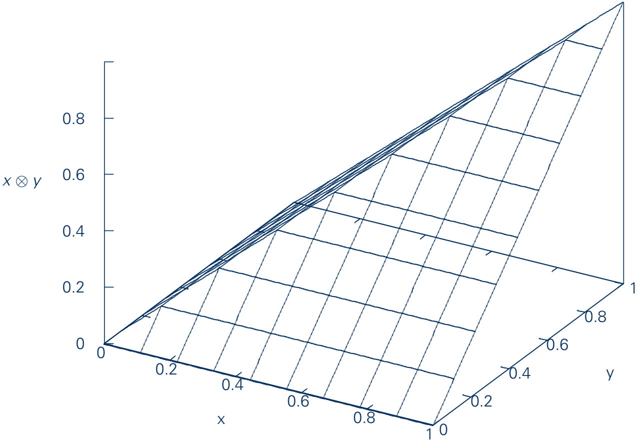
$\otimes$-$\mathcal{SHOI}$ = $\otimes$-$\mathcal{ALC}$ + transitive roles, role hierarchy, nominals, inverse roles

Fundamental continuous t-norms

Gödel: $x \otimes y = \min(x, y)$



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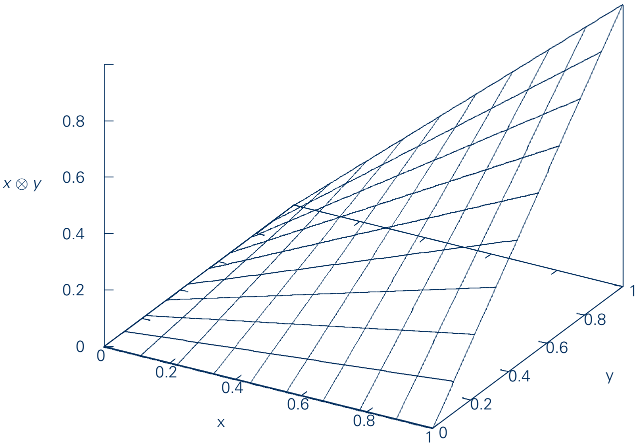

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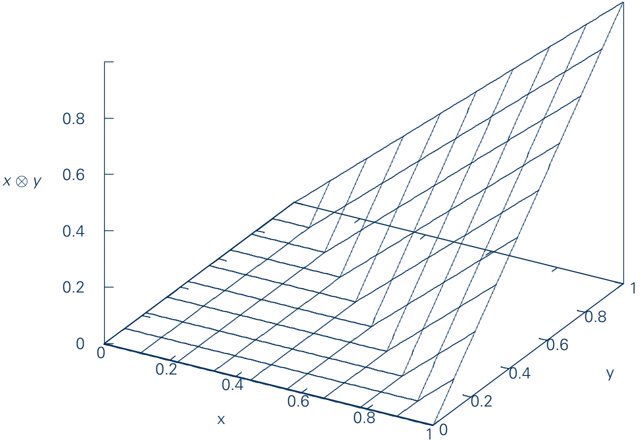
Łukasiewicz: $x \otimes y = \max(0, x + y - 1)$



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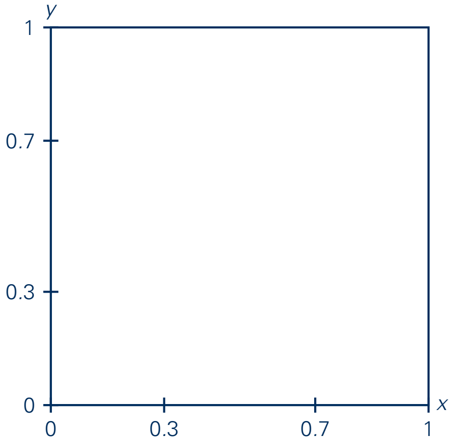

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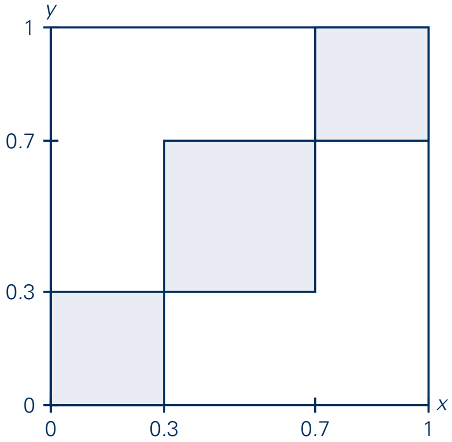

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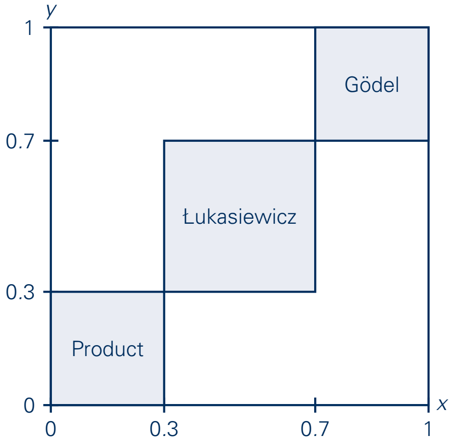

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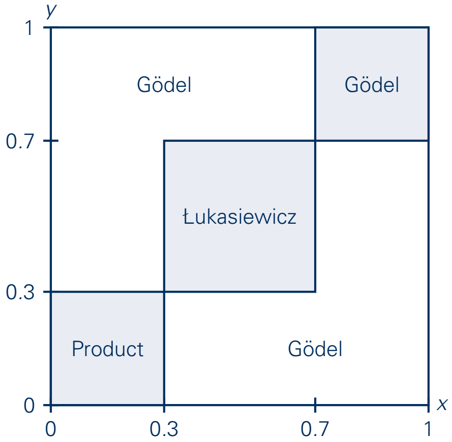

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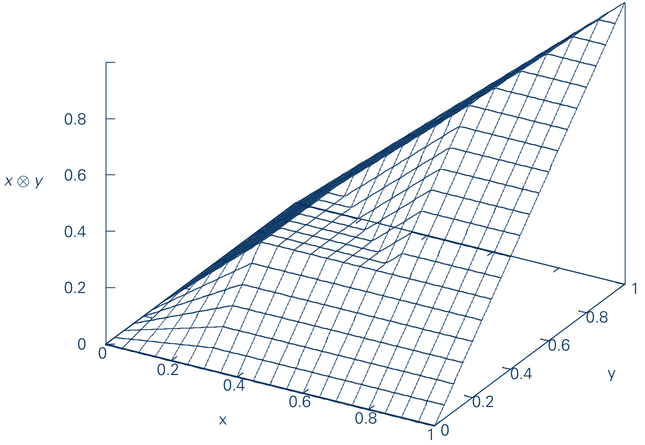

Zero divisors

$\otimes$ has zero divisors if $x \otimes y = 0$ for some $x, y > 0$.

Gödel negation: $\ominus x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

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$\otimes$ does not "start with Łukasiewicz"

Consistency

witnessed interpretation $\mathcal{I}$:
For every role r, concept C, and $x \in \Delta^{\mathcal{I}}$,

$$(\exists r.C)^{\mathcal{I}}(x) = \max_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y) \quad \text{and} \quad (\forall r.C)^{\mathcal{I}}(x) = \min_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y).$$

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(undecidable) (decidable)		constructors $\mathfrak{NEL}$ $[\sqcap, \top, \exists, \neg]$	$\mathfrak{IAL}$ $[\sqcap, \top, \bot, \exists, \forall, \rightarrow]$	$\mathcal{ELC}$ $[\sqcap, \top, \exists, \boxminus]$
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[Baader, Peñaloza 2011], [Cerami, Straccia 2011], [Borgwardt, Peñaloza 2012]
[Bobillo, Delgado, Gómez-Romero, Straccia 2009]

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- $\mathbb{1}(\min\{x \mid x \in X\}) = \min\{\mathbb{1}(x) \mid x \in X\}$ ✓

Crisp models

Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}$ and $x, y \in \Delta^{\mathcal{I}}$.

$$a^{\mathcal{J}} := a^{\mathcal{I}}, \quad A^{\mathcal{J}}(x) := \mathbb{1}\big(A^{\mathcal{I}}(x)\big), \quad r^{\mathcal{J}}(x, y) := \mathbb{1}\big(r^{\mathcal{I}}(x, y)\big)$$

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$\mathcal{O}$ has a model iff crisp($\mathcal{O}$) has a crisp model.

↳ replace every ℓ in $\mathcal{O}$ by $\mathbb{1}(\ell)$

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The fuzzy DL $\otimes$-$\mathcal{SHOI}$ is not fuzzy if $\otimes$ does not have zero divisors!

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Crisp models are not enough!

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Also (ℓ-)satisfiability in $\otimes$-$\mathcal{SHOI}$ is in ExpTime.

However, for entailment (subsumption and instance checking), it is not as easy.

Subsumption: Does $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell$ hold in all models $\mathcal{I}$ of $\mathcal{O}$?

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Consider $\mathcal{O} = \{\langle \top \sqsubseteq A,\ 0.5\rangle\}$, $C = \top$, $D = A$, and $\ell = 1$.

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In every crisp model $\mathcal{J}$ of $\mathcal{O}$, $A^{\mathcal{J}}(x) = 1$ holds.

Conclusions

- consistency/satisfiability w.r.t. witnessed models in $\otimes$-$\mathcal{SHOI}$ is decidable

(undecidable) (decidable)		constructors $\mathfrak{N}\mathcal{EL}$ $[\sqcap, \top, \exists, \neg]$	constructors $\mathfrak{I}\mathcal{AL}$ $[\sqcap, \top, \bot, \exists, \forall, \to]$	constructors $\mathcal{ELC}$ $[\sqcap, \top, \exists, \boxminus]$
assertions	crisp ($\ell = 1$)	Z G	Z G	Π Ł G
assertions	$\geq \ell$	Z G	Z G	$\overline{\text{G}}$ G
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- similar results for general models (without $\forall$)
  [Borgwardt, Distel, Peñaloza DL'12]
- subsumption/instance checking still open
- decidability of consistency still unknown for some fuzzy DLs

Thank You

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