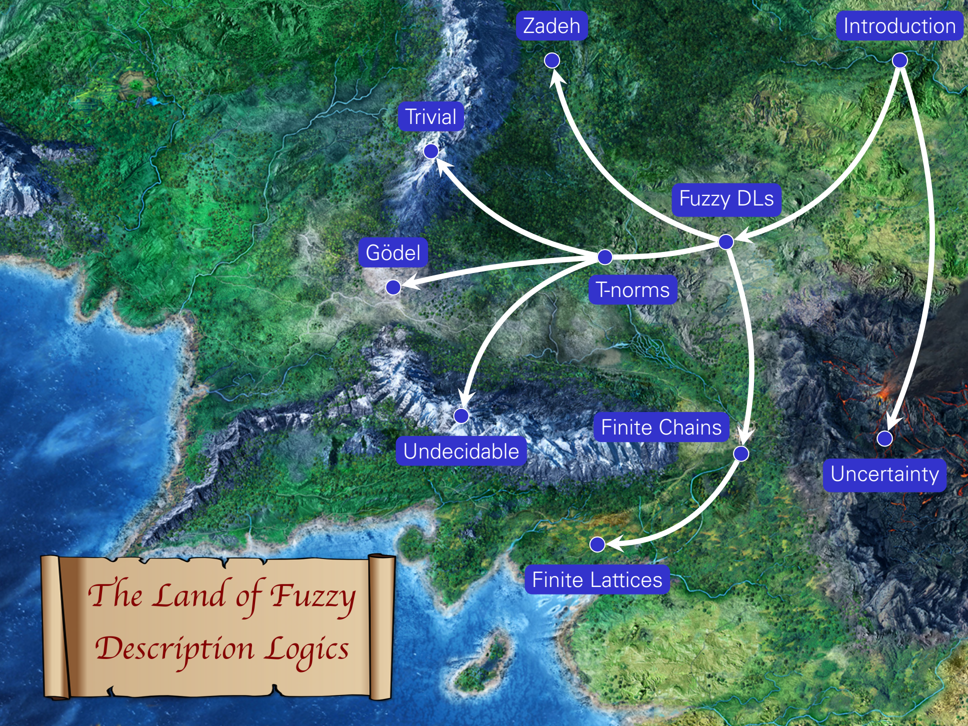
TECHNISCHE UNIVERSITÄT DRESDEN
DRESDEN concept
Institute of Theoretical Computer Science Chair of Automata Theory

A SHORT GUIDE TO FUZZY DESCRIPTION LOGICS

Stefan Borgwardt

Oxford, December 9th, 2014


The Land of Fuzzy
Description Logics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Fuzzy DLs
Gödel
T-norms
Finite Chains
Undecidable
Uncertainty
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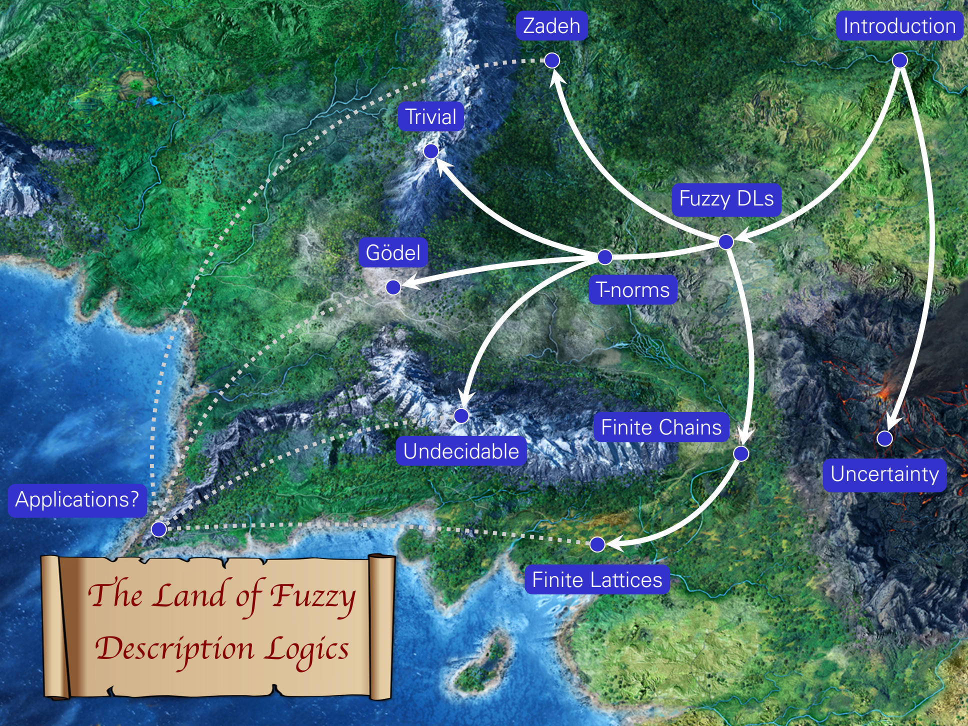
The Land of Fuzzy Description Logics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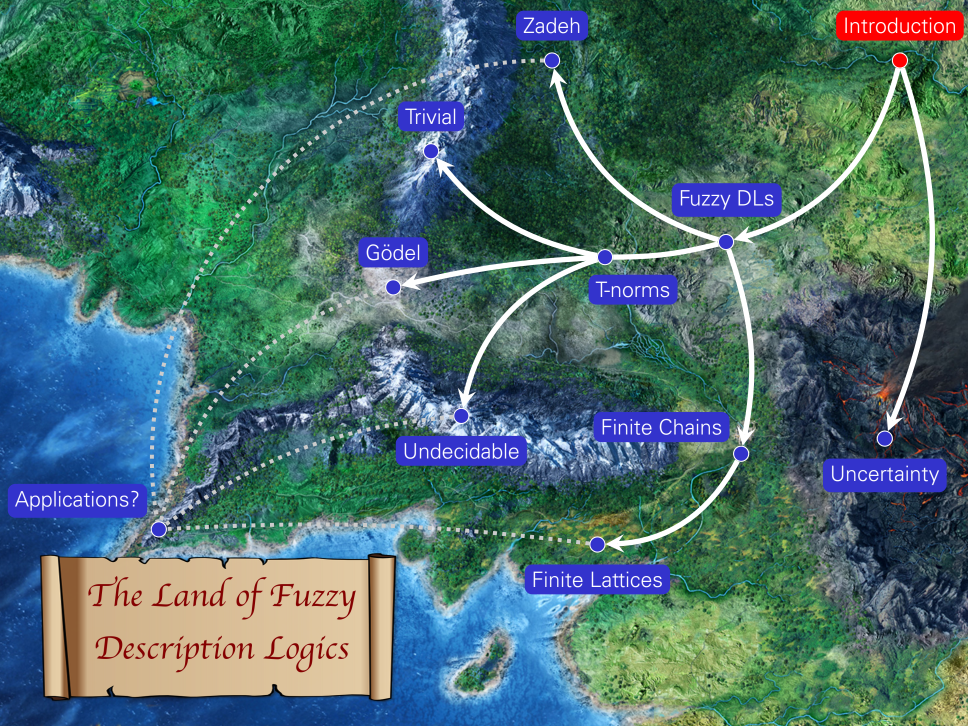
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$$\exists \text{hasDisease.Flu} \sqsubseteq \exists \text{hasSymptom.Headache} \sqcap \exists \text{hasSymptom.Fever}$$

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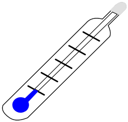

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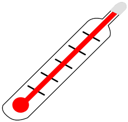

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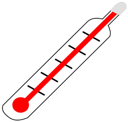

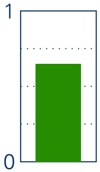

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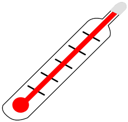

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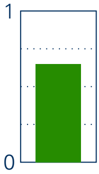

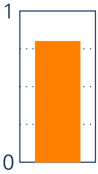

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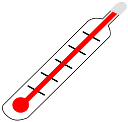

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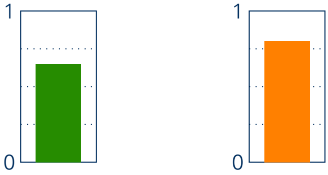

HighConcentration (GALEN),
LowFrequency (SNOMED CT)

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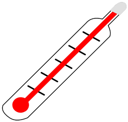

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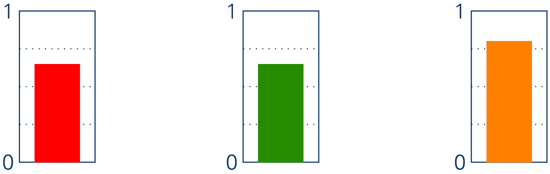

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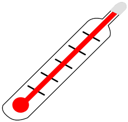

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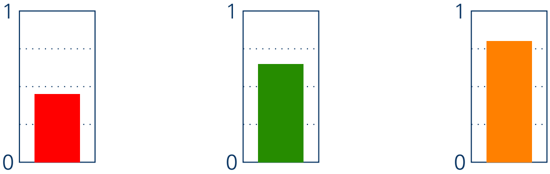

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Uncertainty

$$\exists \text{testResult.Positive} \sqsubseteq \exists \text{hasDisease.Flu}$$

Uncertainty

$\exists \mathsf{testResult}.\mathsf{Positive} \sqsubseteq \exists \mathsf{hasDisease}.\mathsf{Flu}$

...with probability 0.5.

Uncertainty

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. . . with probability 0.5.

Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds

Uncertainty

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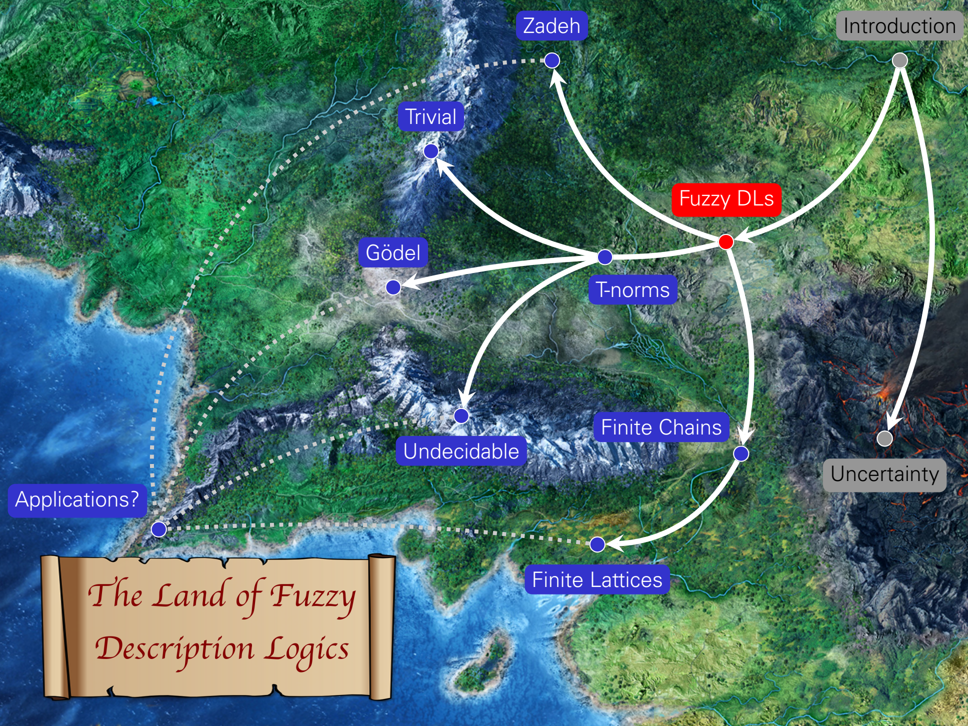
. . . with probability 0.5.

Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds
- $\mathcal{ALCN}$ with possibilistic axioms (Hollunder 1995)
- P-$\mathcal{SHOIN}$(**D**) (Lukasiewicz 2008)
- Prob-$\mathcal{ALC}$ (Lutz and Schröder 2010)
- Bayesian *DL-Lite* and $\mathcal{EL}$ (Ceylan and Peñaloza 2014; d'Amato, Fanizzi, and Lukasiewicz 2008)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Degrees of Truth

$$(L, \otimes, \oplus, \Rightarrow, \ominus)$$

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complete lattice → L

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- L: complete lattice
- $\otimes \colon L \times L \to L$ for conjunction
- $\oplus$: disjunction
- $\Rightarrow$: implication
- $\ominus$: negation

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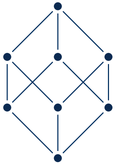

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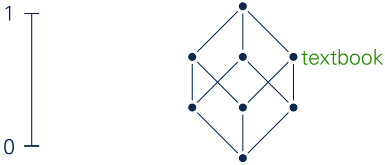

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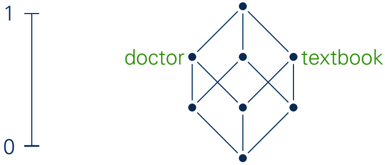

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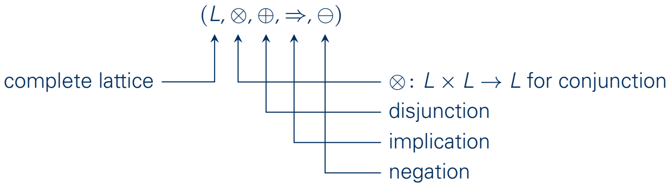

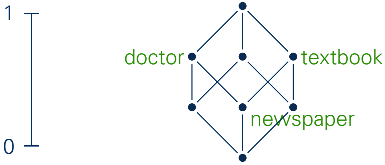

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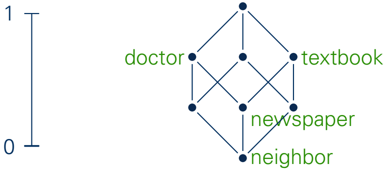

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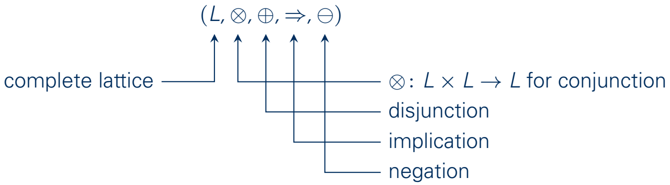

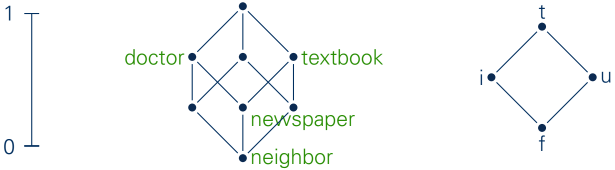

(Belnap 1977; Maier, Ma, and Hitzler 2013)

Fuzzy $\mathcal{ALC}$

$\text{Fever}^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to L \qquad \text{hasSymptom}^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to L \qquad \text{stefan}^{\mathcal{I}} \in \Delta^{\mathcal{I}}$

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$$\{\ldots\} \models \langle \top \sqsubseteq \bot \geq 1\rangle,\ \langle a{:}C \geq p\rangle,\ \langle C \sqsubseteq D \geq p\rangle\ ?$$

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$$\{\dots\} \models \langle \top \sqsubseteq \bot \geq 1\rangle,\ \langle a{:}C \geq p\rangle,\ \langle C \sqsubseteq D \geq p\rangle\ ?$$

best degree? (refers to p in $\langle a{:}C \geq p\rangle$ and $\langle C \sqsubseteq D \geq p\rangle$)

Infinity

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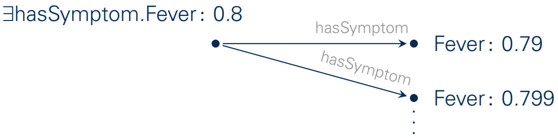

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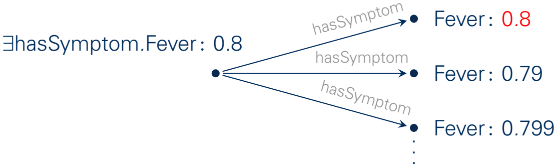

witnessed interpretation
(Hájek 2005)

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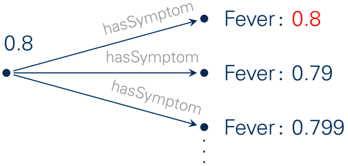
∃hasSymptom.Fever : 0.8



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$$\langle \mathsf{Bulldog} \sqsubseteq \forall \mathsf{hasChild}.\mathsf{Bulldog} \geq 0.5 \rangle$$

Bulldog : 1

Infinity

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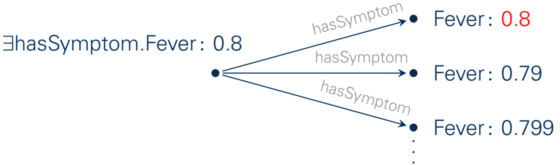

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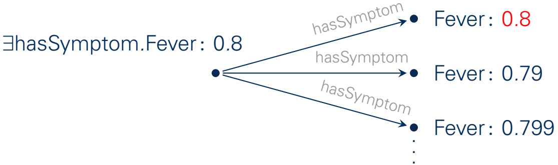

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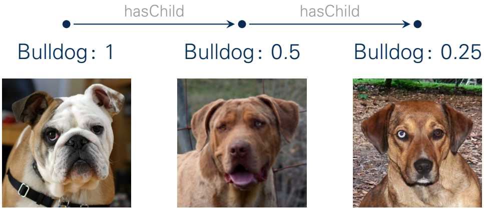

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∃hasSymptom.Fever: 0.8

hasSymptom → Fever: 0.8

hasSymptom → Fever: 0.79

hasSymptom → Fever: 0.799

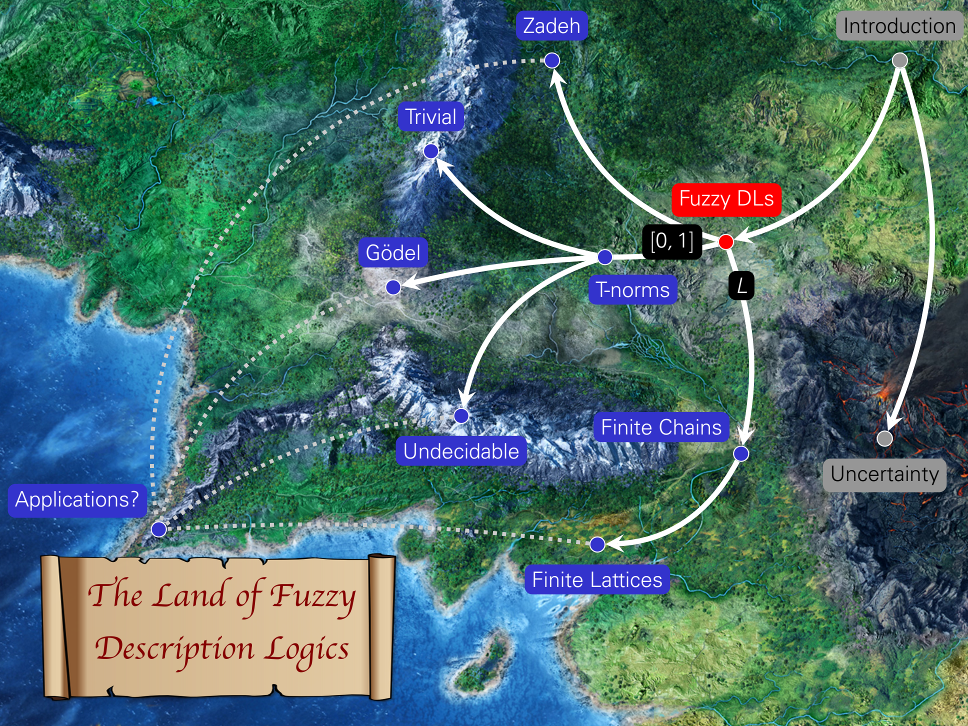
⋮

witnessed interpretation (Hájek 2005)

$\langle \text{Bulldog} \sqsubseteq \forall \text{hasChild.Bulldog} \geq 0.5 \rangle$

Bulldog: 1 —hasChild→ Bulldog: 0.5 —hasChild→ Bulldog: 0.25 —hasChild→ Bulldog: 0.125 ⋯

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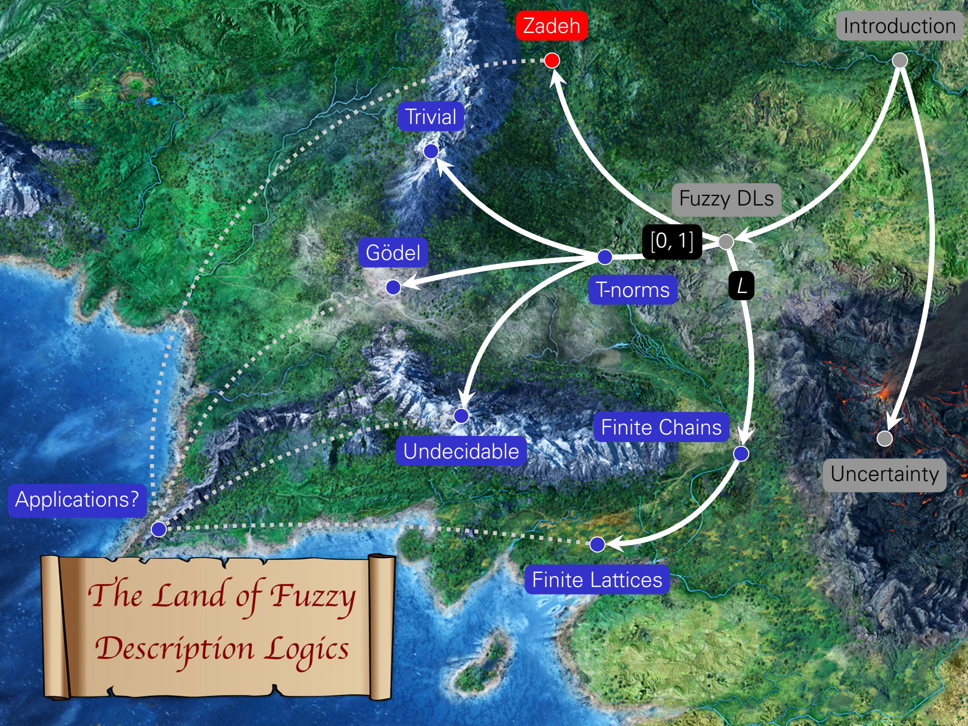
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Finite Chains
Undecidable
Uncertainty
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Finite Lattices
The Land of Fuzzy
Description Logics

Zadeh semantics

$(L, \otimes, \oplus, \Rightarrow, \ominus)$ (Zadeh 1965)

- L: $[0, 1]$
- $\otimes$: $\min(x, y)$
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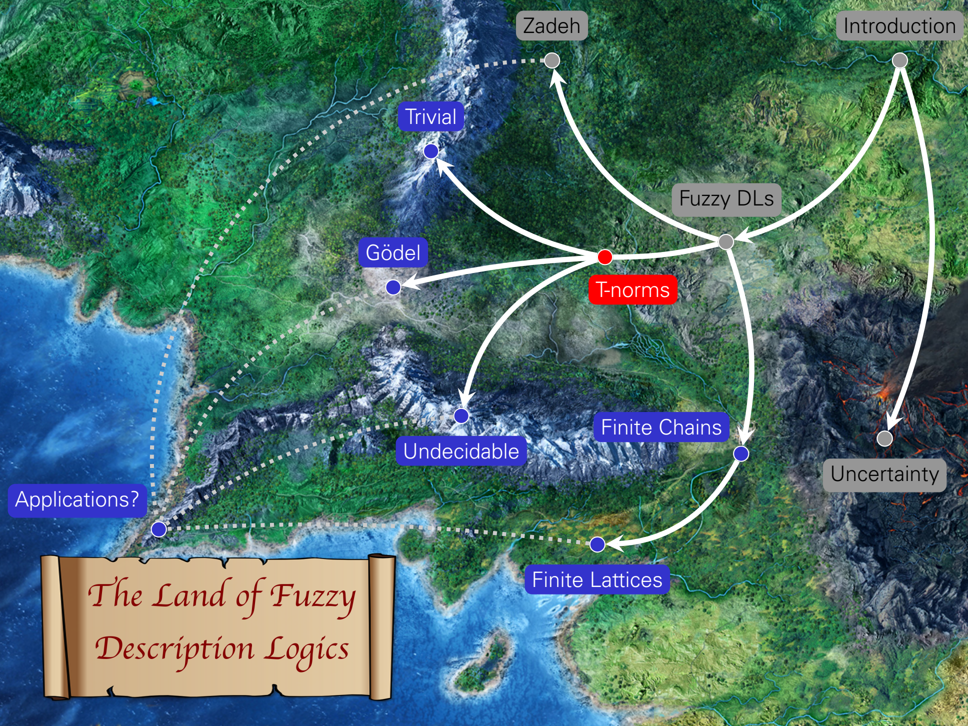

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Triangular Norms

(Hájek 2001)

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[0, 1] → L

t-norm → $\otimes$

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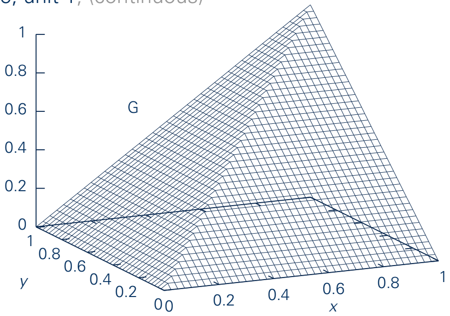

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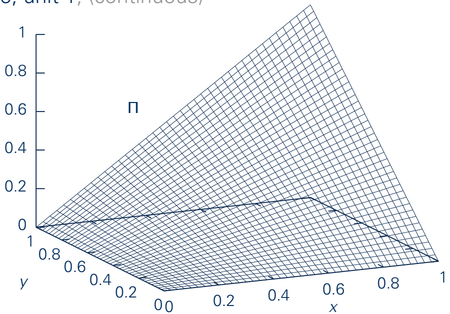

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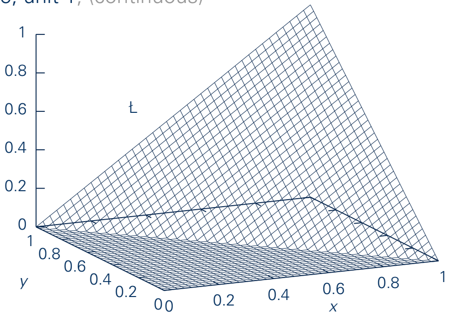

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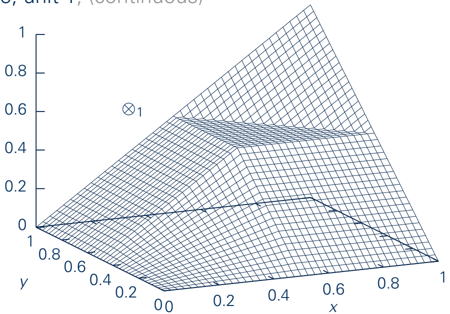

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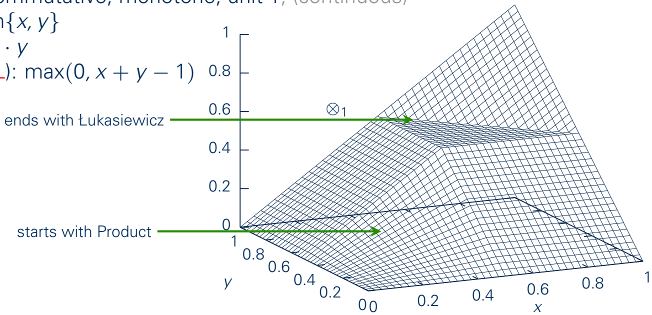

Undecidable Consistency

tableaux algorithm (fuzzyDL): (Bobillo and Straccia 2008)

$$\langle x:(C \sqcap D) \geq p\rangle \quad \rightsquigarrow \quad \langle x:C = v_{x:C}\rangle,\ \langle x:D = v_{x:D}\rangle,\ p \leq v_{x:C} \otimes v_{x:D}$$

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any of the following is bad:

- t-norms starting with Ł
- assertions $\langle a{:}A \bowtie p\rangle$ with $\bowtie \neq \geq$
- involutive negation $1 - x$

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$$\{\langle C \sqsubseteq D \geq 0.7\rangle,\ \langle (a,b):r \geq 0.3\rangle,\ \langle \text{trans}(r) \geq 0.9\rangle,\ \dots\} \text{ is consistent}$$

iff

$$\{C \sqsubseteq D,\ (a,b):r,\ \text{trans}(r),\ \dots\} \text{ is (classically) consistent}$$

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finitely-valued model property $\equiv$ decidable ?

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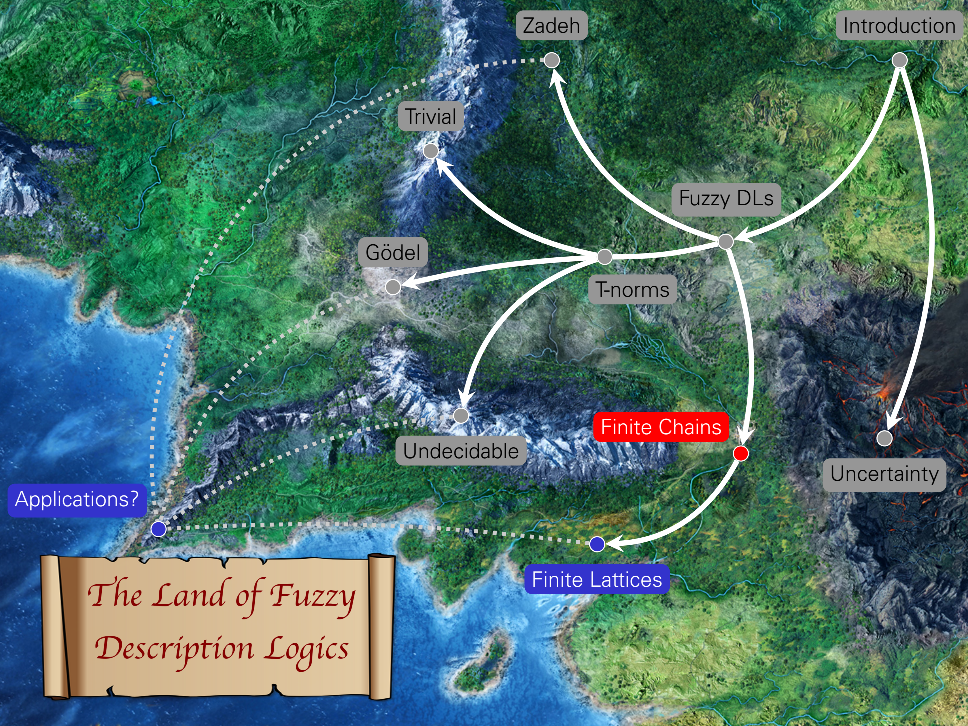
- too weak for undecidability
- ExpTime-complete (Borgwardt, Distel, and Peñaloza 2014)

Triangular Norms

undecidable fuzzy DLs		residual negation			involutive negation	
		$\mathcal{NEL}$	$\mathcal{IEL}$	$\mathcal{SROIQ}$	$\mathcal{ELC}$	$\mathcal{IALC}$
assertions	crisp	starts with Ł	starts with Ł	starts with Ł	Π, Ł	Π, starts with Ł
	$\geq$	starts with Ł	starts with Ł	starts with Ł	not G	not G
	$=$	starts with Ł	not G	not G	not G	not G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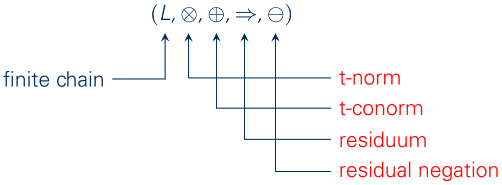

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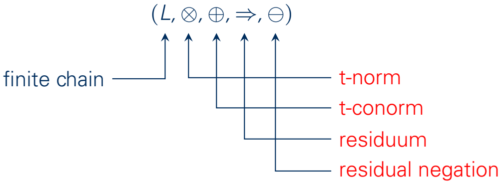

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Finite Chains

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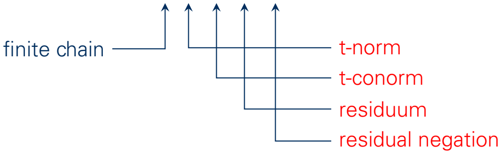

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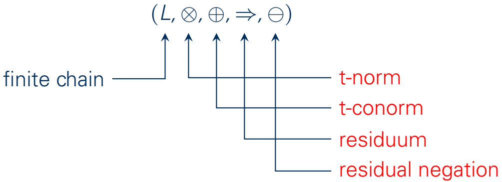

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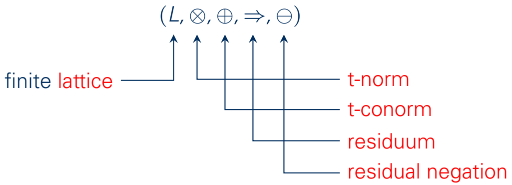
- exponential blow-up (except for Zadeh semantics)
- implemented for f_{KD}-$\mathcal{SROIQ}$ and $\mathrm{G_n}$-$\mathcal{SROIQ}$ in DeLorean

(Bobillo, Delgado, et al. 2012)

Finite Lattices



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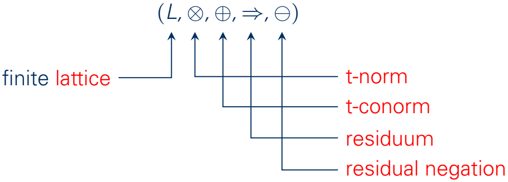

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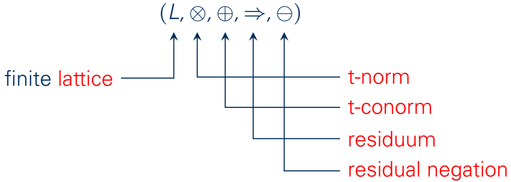

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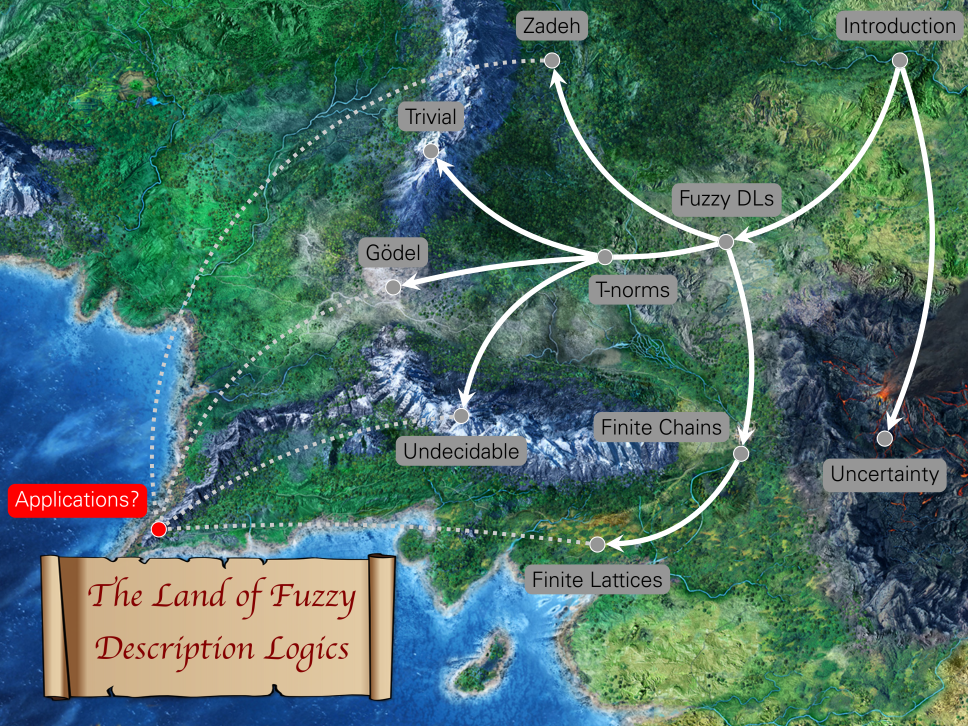
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- using fuzzy degrees consistently:
  - derive from other data
  - use few truth values
  - use order axioms: $\langle \mathsf{stefan}{:}\exists \mathsf{hasDisease.Flu} > \mathsf{rafael}{:}\exists \mathsf{hasDisease.Flu} \rangle$

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Thank you

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- Belnap, Nuel D. (1977). "A Useful Four-Valued Logic". In: *Modern Uses of Multiple-Valued Logic*, pages 7–37.
- Bobillo, Fernando, Miguel Delgado, Juan Gómez-Romero, and Umberto Straccia (2012). "Joining Gödel and Zadeh Fuzzy Logics in Fuzzy Description Logics". In: *Int. J. Uncertain. Fuzz.* 20.4, pages 475–508.
- Bobillo, Fernando and Umberto Straccia (2008). "*fuzzyDL*: An Expressive Fuzzy Description Logic Reasoner". In: *Proc. FUZZ-IEEE'08*, pages 923–930.
- — (2011). "Fuzzy Ontology Representation using OWL 2". In: *Int. J. Approx. Reason.* 52.7, pages 1073–1094.
- — (2014). "Optimising Fuzzy Description Logic Reasoners with General Concept Inclusion Absorption". In: *Fuzzy Set. Syst.* In press.
- Borgwardt, Stefan (2014). "Fuzzy DLs over Finite Lattices with Nominals". In: *Proc. DL'14*. Volume 1193. CEUR-WS, pages 58–70.
- Borgwardt, Stefan, Felix Distel, and Rafael Peñaloza (2012). "How Fuzzy is my Fuzzy Description Logic?" In: *Proc. IJCAR'12*. Volume 7364. LNAI, pages 82–96.
- — (2014). "Decidable Gödel Description Logics without the Finitely-Valued Model Property". In: *Proc. KR'14*, pages 228–237.
- Borgwardt, Stefan, José A. Leyva Galano, and Rafael Peñaloza (2014). "The Fuzzy Description Logic G-$\mathcal{FL}_0$ with Greatest Fixed-Point Semantics". In: *Proc. JELIA'14*. Volume 8761. LNAI, pages 62–76.
- Borgwardt, Stefan and Rafael Peñaloza (2012). "Undecidability of Fuzzy Description Logics". In: *Proc. KR'12*, pages 232–242.

References II

- Borgwardt, Stefan and Rafael Peñaloza (2013a). "Positive Subsumption in Fuzzy $\mathcal{EL}$ with General t-norms". In: *Proc. IJCAI'13*, pages 789–795.
- — (2013b). "The Complexity of Lattice-Based Fuzzy Description Logics". In: *J. Data Semant.* 2.1, pages 1–19.
- Ceylan, İsmail İlkan and Rafael Peñaloza (2014). "Tight Complexity Bounds for Reasoning in the Description Logic $\mathcal{BEL}$". In: *Proc. JELIA'14*. Volume 8761. LNCS, pages 77–91.
- d'Amato, Claudia, Nicola Fanizzi, and Thomas Lukasiewicz (2008). "Tractable Reasoning with Bayesian Description Logics". In: *Proc. SUM'08*. Volume 5291. LNCS, pages 146–159.
- Hájek, Petr (2001). *Metamathematics of Fuzzy Logic (Trends in Logic)*.
- — (2005). "Making Fuzzy Description Logic more General". In: *Fuzzy Set. Syst.* 154.1, pages 1–15.
- Hollunder, Bernhard (1995). "An Alternative Proof Method for Possibilistic Logic and its Application to Terminological Logics". In: *Int. J. Approx. Reason.* 12.2, pages 85–109.
- Lisi, Francesca A. and Umberto Straccia (2014). "A FOIL-Like Method for Learning under Incompleteness and Vagueness". In: *Inductive Logic Programming 2013. Revised Selected Papers.* Volume 8812. LNAI, pages 123–139.
- Lukasiewicz, Thomas (2008). "Expressive Probabilistic Description Logics". In: *Artif. Intell.* 172.6–7, pages 852–883.
- Lutz, Carsten and Lutz Schröder (2010). "Probabilistic Description Logics for Subjective Uncertainty". In: *Proc. KR'10*, pages 393–403.
- Maier, Frederick, Yue Ma, and Pascal Hitzler (2013). "Paraconsistent OWL and Related Logics". In: *Semantic Web* 4.4, pages 395–427.

References III

- Mailis, Theofilos, Rafael Peñaloza, and Anni-Yasmin Turhan (2014). "Conjunctive Query Answering in Finitely-Valued Fuzzy Description Logics". In: *Proc. RR'14*. Volume 8741. LNCS, pages 124–139.
- Mailis, Theofilos, Giorgos Stoilos, Nikolaos Simou, Giorgos B. Stamou, and Stefanos Kollias (2012). "Tractable Reasoning with Vague Knowledge using Fuzzy $\mathcal{EL}^{++}$". In: *J. Intell. Inf. Syst.* 39.2, pages 399–440.
- Mailis, Theofilos, Giorgos Stoilos, and Giorgos Stamou (2010). "Expressive reasoning with Horn rules and fuzzy Description Logics". In: *Knowledge and Information Systems* 25.1, pages 105–136.
- Pan, Jeff Z., Giorgos B. Stamou, Giorgos Stoilos, Stuart Taylor, and Edward Thomas (2008). "Scalable Querying Services over Fuzzy Ontologies". In: *Proc. WWW'08*, pages 575–584.
- Stoilos, Giorgos, Nikos Simou, Giorgos Stamou, and Stefanos Kollias (2006). "Uncertainty and the Semantic Web". In: *IEEE Intell. Syst.* 21.5, pages 84–87.
- Straccia, Umberto (2001). "Reasoning within Fuzzy Description Logics". In: *J. Artif. Intell. Res.* 14, pages 137–166.
- — (2006). "Answering Vague Queries in Fuzzy DL-Lite". In: *Proc. IPMU'06*, pages 2238–2245.
- Tsatsou, Dorothea, Stamatia Dasiopoulou, Ioannis Kompatsiaris, and Vasileios Mezaris (2014). "LiFR: A Lightweight Fuzzy DL Reasoner". In: *The Semantic Web: ESWC 2014 Satellite Events*. Volume 8798. LNCS, pages 263–267.
- Zadeh, Lotfi A. (1965). "Fuzzy Sets". In: *Inform. Control* 8.3, pages 338–353.