TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institut für Theoretische Informatik Lehrstuhl für Automatentheorie

FUZZY DESCRIPTION LOGICS WITH GENERAL CONCEPT INCLUSIONS

Stefan Borgwardt

Tautewalde, Sep 11, 2013

Motivation

laura:Human
laura:Female
laura:Happy
$(\mathrm{laura}, 1)$:has-age
laura:∃wears.Hat
laura:∃has-symptom.Cough
laura:∃has-disease.Cold
Human ⊓ ∃has-age.(≤12) ⊑ Child
Child ⊓ Female ⊑ Girl

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... holds to degree / probability / possibility 0 ... 0.1 ... 0.5 ... 0.9 ... 1

Uncertainty and Vagueness

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- two degrees of truth (*false*, *true*) are replaced by $[0, 1]$ (Zadeh 1965)
- fuzzy sets $A \colon \Delta \to [0, 1]$ instead of sets $A \colon \Delta \to \{0, 1\}$
- conjunction, etc. are interpreted by appropriate truth functions

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- involutive negation $\sim x = 1 - x$
- t-conorm $x \oplus y = \sim(\sim x \otimes \sim y)$

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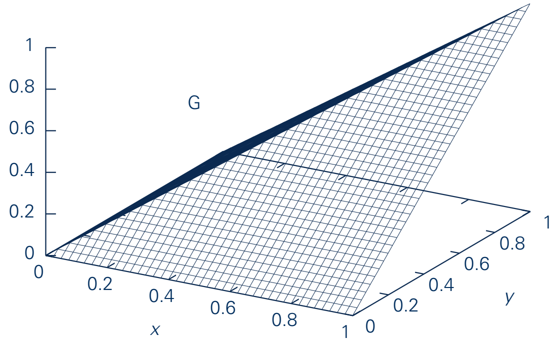

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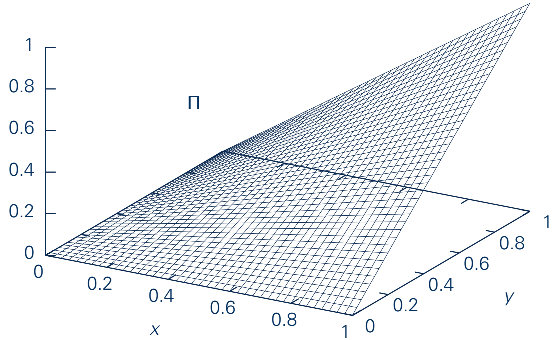

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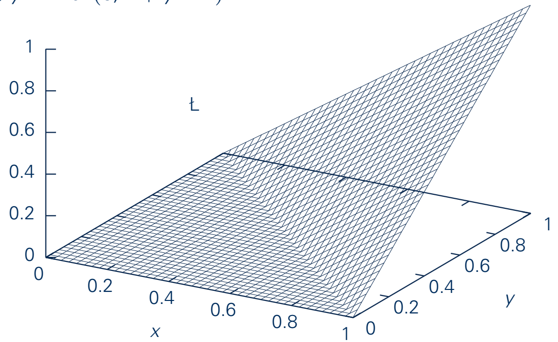

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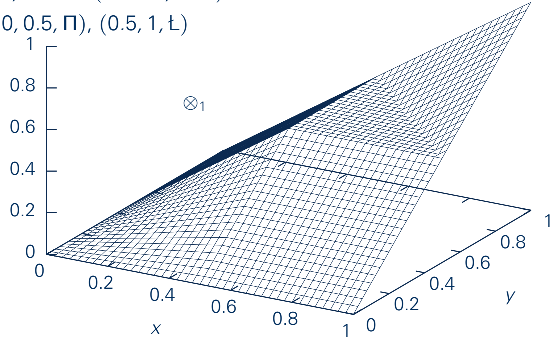

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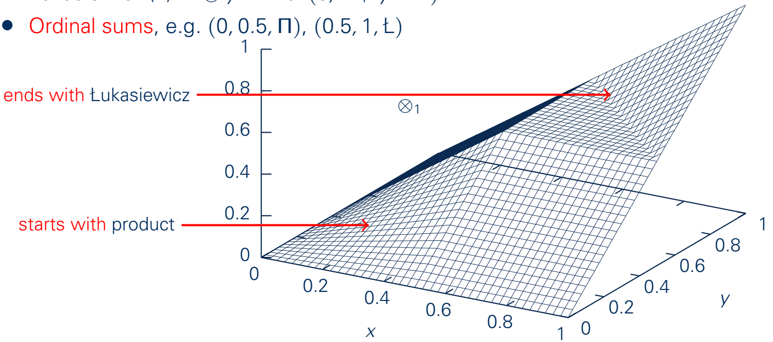

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Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

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- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$
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More constructors:

- $\mathcal{AL} = \mathcal{EL} + \forall r.C$
- $\mathcal{C}$: involutive negation $\neg C$
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$\otimes$-$\mathfrak{I}\mathcal{ALC}$ $\quad$ Π-$\mathfrak{N}\mathcal{EL}$ $\quad$ Ł-$\mathcal{ELC}$

Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a{:}C \rhd p \rangle$: $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd p$
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Applications: (Ciaramella et al. 2010; Meghini, Sebastiani, and Straccia 2001)

- recommender systems with background knowledge
- information retrieval, query relaxation

Tableau Algorithms

Tableau algorithm for $\otimes$-$\mathcal{ALC}$ without GCIs: (Bobillo and Straccia 2009)

$$\langle a{:}C \geq p\rangle \quad \rightsquigarrow \quad \langle a{:}C = v_{a:C}\rangle,\ v_{a:C} \geq p$$

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- NExpTime for Ł-$\mathcal{ALC}$, ExpSpace for Π-$\mathcal{ALC}$
- possible for any finite ordinal sum

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\end{aligned}$$

$$0.5 \cdot \left(\frac{10}{9}\right)^i \leq v_{x_i:A} \quad \rightsquigarrow \quad v_{x_7:A} > 1 \ !$$

Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r.\top \geq 1 \rangle$ can lead to cycles in the tableau
- blocking condition for Π-$\mathcal{ALC}$ with GCIs: (Bobillo and Straccia 2007)
  x is blocked by y if their assertions and constraints are isomorphic
- the algorithm is not sound
  (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)

$$\langle a{:}A \geq 0.5 \rangle,\ \langle A \sqsubseteq \exists r.A \geq 1 \rangle,\ \langle \top \sqsubseteq \neg\exists r.\top \geq 0.1 \rangle$$

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- undecidability results for variants of Π-$\mathcal{ALC}$ and Ł-$\mathcal{ALC}$ with GCIs
  (Baader and Peñaloza 2011a,b; Cerami and Straccia 2011)

(Un)Decidable Fuzzy DLs with GCIs

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Consistency is undecidable (with crisp GCIs) in ... (Borgwardt and Peñaloza 2012)

- ... $\otimes$-$\mathfrak{NEL}$ with crisp assertions if $\otimes$ starts with Ł
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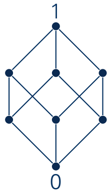
undecidable fuzzy DLs		constructors		
		$\mathcal{NEL}$ $[\sqcap, \top, \exists, \boxminus]$	$\mathcal{IAL}$ $[\sqcap, \top, \bot, \exists, \forall, \rightarrow]$	$\mathcal{ELC}$ $[\sqcap, \top, \exists, \neg]$
assertions	crisp	starts with Ł	starts with Ł	Π Ł
	$\geq$	starts with Ł	starts with Ł	not G
	$=$	starts with Ł	not G	not G

Complete Residuated De Morgan Lattices

More general:

- complete distributive lattice $(L, \vee, \wedge, 0, 1)$



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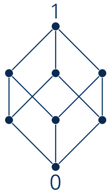

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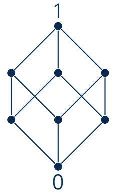

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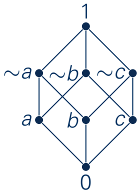

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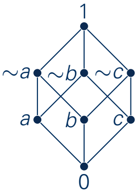

L-$\mathcal{EL}$:

- $\mathsf{Happy}^\mathcal{I}\colon \Delta^\mathcal{I} \to L$
- $\mathsf{likes}^\mathcal{I}\colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to L$
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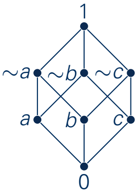

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$\langle(\mathrm{laura}, \mathrm{elisabeth}) : \text{likes} = a\rangle, \langle(\mathrm{laura}, \mathrm{stefan}) : \text{likes} = b\rangle$

Reasoning over Finite Lattices

Reduction to classical reasoning for L-$\mathcal{ALC}$ with GCIs over finite total orders L:

(Bobillo, Delgado, et al. 2012; Straccia 2006)

- introduce cut-concepts and -roles A_p, r_p for every $p \in L$
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Satisfiability and consistency are ExpTime-complete in L-$\mathfrak{I}\mathcal{ALC}$ over finite L:
(Borgwardt and Peñaloza 2013a,c)

- combination of automata construction and tableaux rules
- PSpace-complete without GCIs

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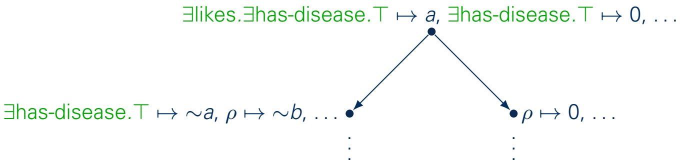

Automata-Based Approach for Satisfiability

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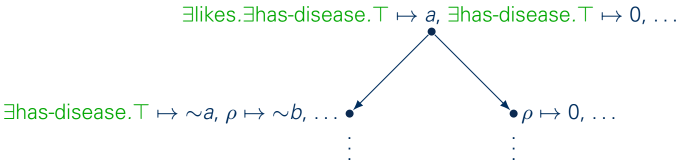
- recognize tree-shaped models by looping tree automaton of exponential size
- satisfiability PSPACE-complete without GCIs
- not suited for consistency



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- try all pre-completions of the assertions (Hollunder 1996)

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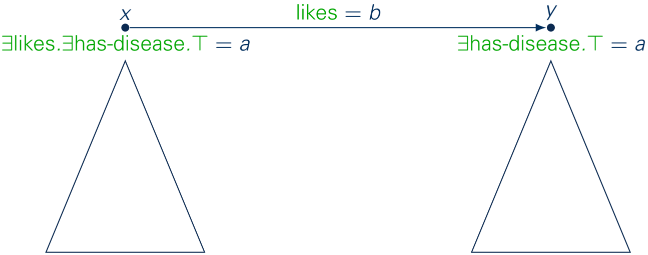

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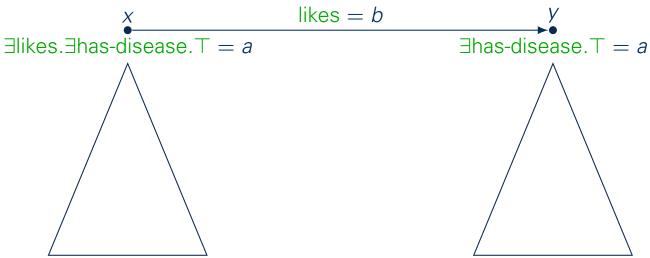

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Thank you!

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