FUZZY DESCRIPTION LOGICS WITH GENERAL CONCEPT INCLUSIONS

Stefan Borgwardt

Tautewalde, Sep 11, 2013
Motivation

laura:Human
laura:Female
laura:Happy
(laura, 1):has-age
laura:∃wears.Hat
laura:∃has-symptom.Cough
laura:∃has-disease.Cold
Human ⊓ ∃has-age.(≤12) ⊑ Child
Child ⊓ Female ⊑ Girl
Motivation

laura: Human
laura: Female
laura: Happy
(laura, 1): has-age
laura: \exists wears.Hat
laura: \exists has-symptom.Cough
laura: \exists has-disease.Cold
Human \sqcap \exists has-age.(\leq 12) \sqsubseteq Child
Child \sqcap Female \sqsubseteq Girl

... holds to degree / probability / possibility 0 ... 0.1 ... 0.5 ... 0.9 ... 1
Uncertainty and Vagueness

\[ \text{Laura: } \exists \text{wears.Hat} \]
Uncertainty and Vagueness

\[
\text{laura} : \exists \text{wears.Hat}
\]

Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds

\[
\text{Prob}-\text{SHOIN}(D) \quad (\text{Lukasiewicz 2008})
\]

\[
\text{Prob-ALC} \quad (\text{Lutz and Schröder 2010})
\]

\[
\text{ALCN} \text{ with possibilistic axioms} \quad (\text{Hollunder 1995})
\]

Fuzzy DLs: (Straccia 2001; Tresp and Molitor 1998; Yen 1991)

- two degrees of truth (false, true) are replaced by \([0, 1]\) (Zadeh 1965)
- fuzzy sets \(A: \Delta \rightarrow [0, 1]\) instead of sets \(A: \Delta \rightarrow \{0, 1\}\)
- conjunction, etc. are interpreted by appropriate truth functions
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Probabilistic/Possibilistic DLs:
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- Prob-\text{ALC} \hspace{1cm} (Lutz and Schröder 2010)
- \text{ALCN} with possibilistic axioms \hspace{1cm} (Hollunder 1995)

Tautewalde, Sep 11, 2013 Fuzzy Description Logics with GCIs
Uncertainty and Vagueness

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$laura: \exists wears.Hat$

$laura: Happy, (laura, elisabeth): likes$
Uncertainty and Vagueness

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Mathematical Fuzzy Logic

- t-norm $\otimes$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$: associative, commutative, monotone, unit 1, (continuous)
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- residuum $\Rightarrow: [0, 1] \times [0, 1] \to [0, 1]$: $x \otimes y \leq z$ iff $y \leq x \Rightarrow z$

- residual negation $\ominus x = x \Rightarrow 0$
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- **residual negation** $\ominus x = x \Rightarrow 0$

- **involutive negation** $\sim x = 1 - x$

- **t-conorm** $x \oplus y = \sim(\sim x \otimes \sim y)$
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Continuous t-norms:
- Gödel (G): $x \otimes y = \min\{x, y\}$

![Graph of continuous t-norms](image)
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- **Ordinal sums**, e.g. \((0, 0.5, \Pi), (0.5, 1, \mathcal{L})\)
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Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$:

- concept names: $\text{Happy}^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]$
- role names: $\text{likes}^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]$
- individual names: $\text{laura}^\mathcal{I} \in \Delta^\mathcal{I}$
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$\mathcal{EL}$:

- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$
- top $\top^\mathcal{I}(x) = 1$
- existential restriction $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$
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More constructors:

- $\mathcal{AL} = \mathcal{EL} + \forall r.C$
- $C$: involutive negation $\neg C$
- $\mathcal{N}$: residual negation $\Box C$
- $\mathcal{I}$: implication $C \rightarrow D$, bottom $\bot$
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$\otimes\text{-\text{IALC}}$ $\Pi\text{-\text{NELC}}$ $\text{L-ELC}$
Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a : C \triangleright p \rangle$: $C^\mathcal{I}(a^\mathcal{I}) \triangleright p$
- role assertion $\langle (a, b) : r \triangleright p \rangle$: $r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \triangleright p$
- GCI $\langle C \sqsubseteq D \geq p \rangle$: $C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geq p$ for all $x \in \Delta^\mathcal{I}$
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\[
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crisp if \( p = 1 \)
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$$p \otimes C^I(x) \leq D^I(x)$$

Witnessed interpretations:

$$(\exists r. C)^I(x) = \max_{y \in \Delta^I} r^I(x, y) \otimes C^I(y).$$

(Hájek 2005)
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Reasoning tasks:

- ontology consistency:
  Does $\mathcal{O}$ have a (witnessed) model?
- concept satisfiability:
  Is there a (witnessed) model $\mathcal{I}$ of $\mathcal{O}$ with $C^\mathcal{I}(x) \geq p$ for some $x \in \Delta^\mathcal{I}$?
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Applications: (Ciaramella et al. 2010; Meghini, Sebastiani, and Straccia 2001)

- recommender systems with background knowledge
- information retrieval, query relaxation

Crisp if $p = 1$
Tableau Algorithms

Tableau algorithm for $\otimes$-$\mathcal{ALC}$ without GCIs: (Bobillo and Straccia 2009)

$\langle a : C \geq p \rangle \leadsto \langle a : C = v_a : C \rangle, v_a : C \geq p$

$\langle (a, b) : r \geq p \rangle \leadsto \langle (a, b) : r = v_{(a,b)} : r \rangle, v_{(a,b)} : r \geq p$
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$\langle (a,b):r \geq p \rangle \leadsto \langle (a,b):r = v_{(a,b):r}, v_{(a,b):r} \geq p \rangle$

$\langle x:C \cap D = v \rangle \leadsto \langle x:C = v_{x:C}, x:D = v_{x:D}, v = v_{x:C} \otimes v_{x:D} \rangle$

$\ldots$

$\text{Deterministic exponential time}$

$O$ is consistent iff the constraints have a solution (NP-hard)

$\mathcal{NEXP} \subseteq \mathcal{XPTIME}$ for $\mathcal{Ł\text{-}ALC}$, $\mathcal{E\text{-}XP} \subseteq \mathcal{PACE}$ for $\mathcal{Π\text{-}ALC}$

$\text{Possible for any finite ordinal sum}$
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$$\langle x: C \cap D = v \rangle \leadsto \langle x: C = v_{x:C} \rangle, \langle x: D = v_{x:D} \rangle, \; v = v_{x:C} \otimes v_{x:D}$$

$$\ldots$$

$$\langle x: \exists r. C = v \rangle \leadsto \langle (x, y): r = v_{(x,y):r} \rangle, \langle y: C = v_{y:C} \rangle, \; v = v_{(x,y):r} \otimes v_{y:C}$$

$$\langle x: \exists r. C = v \rangle, \langle (x, y): r = v' \rangle \leadsto \langle y: C = v_{y:C} \rangle, \; v \geq v' \otimes v_{y:C}$$

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\langle x: C \cap D = v \rangle \leadsto \langle x: C = v_{x: C} \rangle, \langle x: D = v_{x: D} \rangle, \ v = v_{x: C} \otimes v_{x: D} \\
\ldots
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\langle x: \exists r.C = v \rangle \leadsto \langle (x, y): r = v_{(x, y): r} \rangle, \langle y: C = v_{y: C} \rangle, \ v = v_{(x, y): r} \otimes v_{y: C} \\
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- deterministic exponential time
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\[ \langle x : C \sqcap D = v \rangle \leadsto \langle x : C = v_x : C \rangle, \langle x : D = v_x : D \rangle, \ v = v_x : C \otimes v_x : D \]
\[ \ldots \]
\[ \langle x : \exists r.C = v \rangle \leadsto \langle (x, y) : r = v_{(x, y)} : r \rangle, \langle y : C = v_y : C \rangle, \ v = v_{(x, y)} : r \otimes v_y : C \]
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\[ \ldots \]

- deterministic exponential time
- $\mathcal{O}$ is consistent iff the constraints have a solution (NP-hard)
- $\text{NEXPTIME}$ for $\mathcal{L}$-$\mathcal{ALC}$, $\text{EXPSPACE}$ for $\Pi$-$\mathcal{ALC}$
- possible for any finite ordinal sum
Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- blocking condition for $\Pi$-ALC with GCIs: (Bobillo and Straccia 2007)
  $x$ is blocked by $y$ if their assertions and constraints are isomorphic
Fuzzy GCIs

- GCIs like $\langle T \subseteq \exists r.T \geq 1 \rangle$ can lead to cycles in the tableau.
- Blocking condition for $\Pi$-$ALC$ with GCIs: $x$ is blocked by $y$ if their assertions and constraints are isomorphic.
- The algorithm is not sound.

(Bobillo and Straccia 2007)

(Cerami and Straccia 2011)

(Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)
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\[
\langle a : A \geq 0.5 \rangle, \langle A \sqsubseteq \exists r. A \geq 1 \rangle, \langle \top \sqsubseteq \neg \exists r. \top \geq 0.1 \rangle
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- blocking condition for $\Pi$-ALC with GCIs: \cite{Bobillo2007} $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound \cite{Baader2011a,Bobillo2011a,Bobillo2011b}

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\[
0.5 \leq v_{a : A}, \quad v_{a : A} \leq v_{x_1 : A} \cdot v_{(a,x_1) : r}, \quad v_{(a,x_1) : r} \leq 0.9,
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v_{x_1:A} \leq v_{x_2:A} \cdot v_{(x_1,x_2):r}, \quad v_{(x_1,x_2):r} \leq 0.9,
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(Bobillo and Straccia 2007)

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$$v_{x_2:A} \leq v_{x_3:A} \cdot v_{(x_2,x_3):r}, \quad v_{(x_2,x_3):r} \leq 0.9,$$

$$0.5 \cdot \left( \frac{10}{9} \right)^i \leq v_{x_i:A} \xrightarrow{\sim} v_{x_7:A} > 1 !$$
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- blocking condition for $\Pi$-\textit{ALC} with GCIs: $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound

$$\langle a:A \geq 0.5 \rangle, \langle A \sqsubseteq \exists r.A \geq 1 \rangle, \langle T \sqsubseteq \neg \exists r.T \geq 0.1 \rangle$$

$$0.5 \leq v_{a:A}, v_a:A \leq v_{x_1:A} \cdot v_{(a,x_1):r}, v_{(a,x_1):r} \leq 0.9,$$

$$v_{x_1:A} \leq v_{x_2:A} \cdot v_{(x_1,x_2):r}, v_{(x_1,x_2):r} \leq 0.9,$$

$$v_{x_2:A} \leq v_{x_3:A} \cdot v_{(x_2,x_3):r}, v_{(x_2,x_3):r} \leq 0.9$$

$$0.5 \cdot \left( \frac{10}{9} \right)^i \leq v_{x_i:A} \implies v_{x_7:A} > 1 !$$

- undecidability results for variants of $\Pi$-\textit{ALC} and $L$-\textit{ALC} with GCIs

(Baader and Peñaloza 2011a, b; Cerami and Straccia 2011)
(Un)Decidable Fuzzy DLs with GCIs

Consistency is undecidable (with crisp GCIs) in $\otimes$-N$\mathcal{E}$L with crisp assertions if $\otimes$ starts with Ł

• $\otimes$-E$\mathcal{L}$C with $\geq$-assertions and $\otimes$-I$\mathcal{A}$L with $=$-assertions for all continuous t-norms except the Gödel t-norm

Consistency is decidable (E$\mathcal{X}$PTIME-complete) in $\otimes$-I$\mathcal{A}$L with $\geq$-assertions if $\otimes$ does not start with Ł (Borgwardt, Distel, and Peñaloza 2012)

• residual negation is crisp: $\neg x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$

• restrict to crisp models of crisp ontologies

undecidable fuzzy DLs

constructors

N$\mathcal{E}$L I$\mathcal{A}$L E$\mathcal{L}$C $\begin{bmatrix} \otimes, \top, \exists, \top \end{bmatrix}$ $\begin{bmatrix} \otimes, \top, \top, \top, \top \end{bmatrix}$ $\begin{bmatrix} \otimes, \top, \top, \top, \neg \end{bmatrix}$ assertions crisp starts with Ł starts with Ł starts with Ł $\Pi$ Ł $\geq$ Ł $=$ Ł not G not G not G
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- restrict to crisp models of crisp ontologies
Complete Residuated De Morgan Lattices

More general:
- complete distributive lattice \((L, \lor, \land, 0, 1)\)

\[
\begin{align*}
\text{Happy} & : \Delta^I \to L \\
\text{likes} & : \Delta^I \times \Delta^I \to L \\
(\exists r.C)^I(x) & = \bigvee_{y \in \Delta^I} r^I(x, y) \land C^I(y)
\end{align*}
\]

\[\langle (laura, elisabeth) : \text{likes} = a \rangle, \langle (laura, stefan) : \text{likes} = b \rangle\]
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- (generalized) t-norm \(\otimes: L \times L \rightarrow L:\)
  associative, commutative, monotone, unit 1, ("continuous")
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- involutive De Morgan negation \(\sim: L \rightarrow L\)
- t-conorm \(x \oplus y = \sim(\sim x \otimes \sim y)\)

\[\begin{array}{c}
\sim a \\
\sim b \\
\sim c \\
1 \\
\end{array}
\begin{array}{c}
a \\
b \\
c \\
0 \\
\end{array}\]
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**L-\(\mathcal{E}\mathcal{L}\):**

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- **likes** \(\mathcal{I}: \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to L\)
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\(<(laura, elisabeth): \text{likes} = a>, <(laura, stefan): \text{likes} = b>\)
Reduction to classical reasoning for $L$-$\mathcal{ALC}$ with GCIs over finite total orders $L$:

- introduce cut-concepts and -roles $A_p, r_p$ for every $p \in L$
- $A_p \doteq$ all individuals $x$ with $A^I(x) \geq p$

(Bohillo, Delgado, et al. 2012; Straccia 2006)
Reasoning over Finite Lattices

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- $\langle a : C \cap D \geq p \rangle \rightsquigarrow a : \bigsqcup_{q \otimes q' = p} C_q \cap D_{q'}$
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- exponential in the size of $\mathcal{O} \leadsto 2\text{-EXP\textsc{TIME}}$
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- exponential in the size of $\mathcal{O} \rightsquigarrow \text{2-EXP\!TIME}$

Satisfiability and consistency are \textsc{EXPTIME}-complete in $L$-\textsc{IALC} over finite $L$:
(Borgwardt and Peñaloza 2013a,c)

- combination of automata construction and tableaux rules
- \textsc{PSPACE}-complete without GCIs
Automata-Based Approach for Satisfiability

- adaptation of classical construction (Baader, Hladik, and Peñaloza 2008)
- recognize tree-shaped models by looping tree automaton of exponential size
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\[
\exists \text{likes.} \exists \text{has-disease.} \top \mapsto a, \exists \text{has-disease.} \top \mapsto 0, \ldots
\]
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∃likes.∃has-disease.⊤ ↦ a, ∃has-disease.⊤ ↦ 0, ...
∃has-disease.⊤ ↦ ¬a, ρ ↦ ¬b, ...
ρ ↦ 0, ...

...
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- recognize tree-shaped models by looping tree automaton of exponential size
- satisfiability PSPACE-complete without GCIs
- not suited for consistency

∃\text{likes}.∃\text{has-disease}.\top \rightarrow a, ∃\text{has-disease}.\top \rightarrow 0, \ldots

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Tableau Algorithm for Consistency

- try all pre-completions of the assertions  
  
(Hollunder 1996)

\[
\langle x: \exists \text{likes}. \exists \text{has-disease}. \top \geq a \rangle, \langle (x, y): \text{likes} = b \rangle
\]

\[
\exists \text{likes}. \exists \text{has-disease}. \top = a \\
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Tableau Algorithm for Consistency

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Summary

- fuzzy DLs with GCIs over \([0, 1]\) often undecidable or trivial
- tight complexity results for fuzzy DLs over finite lattices
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Open questions:
- for some cases decidability still unknown
- in $\mathcal{EL}$, consistency is trivial, but what about subsumption? (Borgwardt and Peñaloza 2013b)
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Thank you!
References I


