TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institut für Theoretische Informatik Lehrstuhl für Automatentheorie

REASONING WITH VAGUENESS IN DESCRIPTION LOGICS USING AUTOMATA AND TABLEAUX

Statusvortrag

Stefan Borgwardt

Dresden, 10.01.2012

Motivation

$$\text{Human}(\mathrm{elisabeth}), \quad \text{Female}(\mathrm{elisabeth}),$$

$$\text{has-age}(\mathrm{elisabeth}, 2), \ \text{has-father}(\mathrm{elisabeth}, \mathrm{stefan}),$$

$$\text{has-friend}(\mathrm{elisabeth}, \mathrm{tabea}), \ \text{Happy}(\mathrm{elisabeth}),$$

$$\text{Human} \sqcap \exists\text{has-age}.(\leq 12) \sqsubseteq \text{Child},$$

$$\text{Child} \sqcap \text{Female} \sqsubseteq \text{Girl}$$

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... holds to degree / probability / possibility 0 ... 0.1 ... 0.5 ... 0.9 ... 1

Outline

- Description Logics
- Uncertainty and Vagueness
- Fuzzy Description Logics
  - Zadeh Semantics
  - t-norm Based Fuzzy Logic
  - Problems with General Concept Inclusions
- Summary

Description Logics (DLs)

$\mathcal{ALC}$ [Baader *et al.*, 2007]:

name	syntax	semantics ($\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$)
concept name	$A \in \mathsf{N_C}$	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
role name	$r \in \mathsf{N_R}$	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
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conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
top, bottom	$\top, \bot$	$\Delta^{\mathcal{I}}, \emptyset$
existential restriction	$\exists r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} \colon (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
value restriction	$\forall r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}} \colon (x, y) \in r^{\mathcal{I}} \to y \in C^{\mathcal{I}}\}$

Knowledge Representation and Reasoning

ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$

Assertional knowledge (ABox $\mathcal{A}$):

- concept assertion $C(a)$: $a^{\mathcal{I}} \in C^{\mathcal{I}}$
- role assertion $r(a, b)$: $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$

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Terminological knowledge (TBox $\mathcal{T}$):

- concept definition $A \equiv C$: $A^\mathcal{I} = C^\mathcal{I}$ $\rightsquigarrow$ acyclic TBox
- general concept inclusion (GCI) $C \sqsubseteq D$: $C^\mathcal{I} \subseteq D^\mathcal{I}$ $\rightsquigarrow$ general TBox

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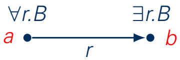
Reasoning tasks:

- ontology consistency: Does $\mathcal{O}$ have a model?
- concept satisfiability: Is there a model of $\mathcal{O}$ in which C is non-empty?

Reasoning Algorithms

Tableau algorithms: $r(a,b),\ (\forall r.B)(a),\ (\exists r.B)(b), \quad B \sqsubseteq \exists r.A,\ A \sqsubseteq \exists r.B$

- rules try to complete the ABox into a model of the TBox



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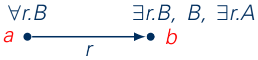

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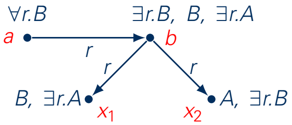

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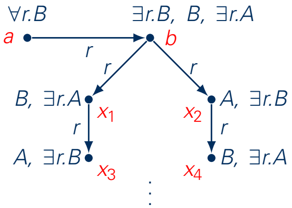

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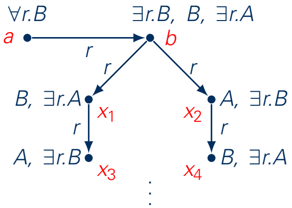

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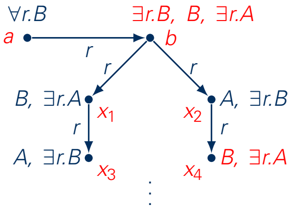

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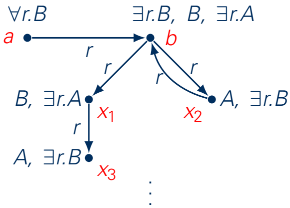

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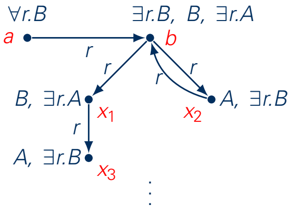

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- $\{r(a,b),\ (\forall r.B)(a),\ (\exists r.B)(b),\ B(b),\ (\exists r.A)(b),\ B(x_1),\ \dots\}$

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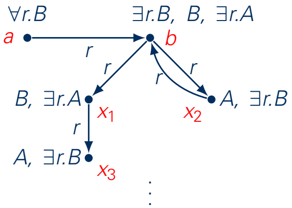

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- disjunction $\sqcup$ introduces nondeterminism
- a clash $(A, \neg A)$ indicates failure of the model construction
- complexity: NExpTime

Outline

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How can we express uncertainty about our knowledge?

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Overview: [Lukasiewicz and Straccia, 2008]

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$\langle \text{has-friend}(\mathrm{elisabeth}, \mathrm{tabea}) \geq 0.5 \rangle$, $\langle \text{Happy}(\mathrm{elisabeth}) \geq 0.8 \rangle$

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Fuzzy Description Logics

syntax	semantics
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$\top$, $\bot$	1, 0
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Unfoldable TBox: acyclic concept definitions $A \equiv C$ as before ($A^\mathcal{I} = C^\mathcal{I}$)

Fuzzy Reasoning

Witnessed interpretation $\mathcal{I}$: [Hájek, 2005]
For every $r \in \mathsf{N_R}$, concept C, and $x \in \Delta^{\mathcal{I}}$,

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- What is the best satisfiability degree ℓ?

Fuzzy Reasoning Algorithms

Tableau algorithm w.r.t. witnessed models and unfoldable TBoxes:
[Tresp and Molitor, 1998; Straccia, 2005]

$$\langle C(a) \geq \ell \rangle \rightsquigarrow \langle C(a) = v_{C(a)} \rangle,\ v_{C(a)} \geq \ell$$

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Tableau algorithm w.r.t. witnessed models and unfoldable TBoxes:
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$$\ldots$$

- deterministic exponential time
- $\mathcal{O}$ is consistent iff the constraints have a solution (NP-complete)
- best degree: maximize $v_{C(x)}$

Fuzzy Reasoning Algorithms (II)

Observation: reasoning is effectively finite-valued [Straccia, 2001]

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Crispify the ontology by introducing cut-concepts and -roles: [Straccia, 2004]

- A_ℓ, r_ℓ for every $\ell \in N^{\mathcal{O}}$
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- $\langle (C \sqcup D)(a) \geq \ell \rangle \rightsquigarrow (C_\ell \sqcup D_\ell)(a)$
- $\langle C \sqsubseteq D \geq \ell \rangle \rightsquigarrow C_{>1-\ell} \sqsubseteq D_\ell$
- polynomial in the size of $\mathcal{O}$

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Fuzzy logic: [Hájek, 2001]

- t-norm $\otimes\colon [0,1] \times [0,1] \to [0,1]$:
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All continuous t-norms are (isomorphic to) ordinal sums of the fundamental t-norms.
[Mostert and Shields, 1957]

Fuzzy Reasoning Algorithms (III)

Tableau algorithm with constraints:

- possible for any finite ordinal sum
- NExpTime for G-$\mathcal{ALC}$ and Ł-$\mathcal{ALC}$ [Straccia and Bobillo, 2007]
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Crispification:

- G-$\mathcal{ALC}$: [Bobillo *et al.*, 2009]
  - reasoning is effectively finite-valued
  - translation to crisp $\mathcal{ALCH}$ is exponential
- different approach: [Bobillo and Straccia, 2011]
  - restrict reasoning a priori to finitely many values
  - for instance Łukasiewicz t-norm on $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \rightsquigarrow$ Ł$_n$-$\mathcal{ALC}$

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- since then several undecidability results for variants of Π-$\mathcal{ALC}$ and Ł-$\mathcal{ALC}$ [Baader and Peñaloza, 2011a,b; Cerami and Straccia, 2011]

Reduction of the Post Correspondence Problem

Given pairs of words $(v_1, w_1), \ldots, (v_p, w_p)$ over Σ with $|\Sigma| > 1$,
is there a non-empty sequence of indices $i_1 \ldots i_k \in \{1, \ldots, p\}^+$ such that
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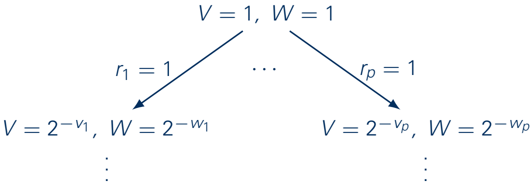

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$$r_1 = 1 \qquad \cdots \qquad r_p = 1$$

$$V = 2^{-v_1},\ W = 2^{-w_1} \qquad\qquad V = 2^{-v_p},\ W = 2^{-w_p}$$

$$\vdots \qquad\qquad\qquad \vdots$$

- $\mathcal{O}$ restricts V and W to be unequal (except at the root)
- $\mathcal{O}$ is consistent iff there is no solution

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Thank you!

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