TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Institute of Theoretical Computer Science Chair of Automata Theory

COMPUTING LOCAL UNIFIERS IN THE DESCRIPTION LOGIC $\mathcal{EL}$ WITHOUT THE TOP CONCEPT

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Wrocław, July 31st, 2011

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$\mathcal{EL}^{-\top}$ (concept name, role name, conjunction, existential restriction)

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- Description logics are used to formulate ontologies
- SNOMED CT is based on $\mathcal{EL}$, but does not use $\top$
- Unification can be used to detect redundancies

Unification in $\mathcal{EL}^{(-\top)}$

Some concept names are variables ($X \in N_v$), all others are constants ($A \in N_c$).

$\rightarrow$ unification problem: $\Gamma = \{C_1 \equiv^? D_1, \ldots, C_n \equiv^? D_n\}$

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A unifier σ substitutes variables with concept terms such that

$$\sigma(C_1) \equiv \sigma(D_1), \ldots, \sigma(C_n) \equiv \sigma(D_n).$$

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Unification modulo the equational theory of bounded semilattices with monotone operators:

A	$\rightarrow$	free constant
X	$\rightarrow$	variable
$\sqcap$	$\rightarrow$	binary associative, commutative, idempotent operator
$\exists r.C$	$\rightarrow$	unary monotone operator
$\top$	$\rightarrow$	constant; unit for $\sqcap$

Previous Results

Unification in $\mathcal{EL}$ is NP-complete:

- Matching is NP-hard [Baader, Küsters 2000].
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Unification in $\mathcal{EL}^{-\top}$ is PSPACE-complete [CADE 2011].

In this talk: Local unifiers in $\mathcal{EL}^{-\top}$ may be of exponential size.

Preliminaries

Atom: concept name or existential restriction

Non-variable atom: concept constant or existential restriction

Flat atom: atom of depth ≤ 1

Flat unification problem:
All equations are of the form $C_1 \sqcap \cdots \sqcap C_n \equiv^? D_1 \sqcap \cdots \sqcap D_m$ for flat atoms $C_1, \ldots, C_n, D_1, \ldots, D_m$.

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- All atoms subsumed by an existential restriction $\exists r.E$ are of the form $\exists r.E'$ with $E' \sqsubseteq E$.
- All concept terms subsumed by a conjunction of atoms $D_1 \sqcap \cdots \sqcap D_m$ are conjunctions of atoms $C_1 \sqcap \cdots \sqcap C_n$ such that for every D_j there is a C_i with $C_i \sqsubseteq D_j$.

$\mathcal{EL}$ vs. $\mathcal{EL}^{-\top}$

Particle: atom of the form $\exists r_1. \ldots . \exists r_n.A$

If C is an $\mathcal{EL}^{-\top}$-concept term and B is a particle, then $B \sqsubseteq C$ implies $B \equiv C$.

Part(C): $\mathrm{Part}(A \sqcap \exists r.(A \sqcap \exists r.B)) = \{A, \exists r.A, \exists r.\exists r.B\}$

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In $\mathcal{EL}$, it suffices to check for local unifiers σ:

$$\sigma(X) = \sigma(D_1) \sqcap \dots \sqcap \sigma(D_m),$$

where $D_1, \dots, D_m$ are non-variable atoms of the unification problem.

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Example:

$$\Gamma : \quad X \equiv^? Y \sqcap A, \quad \exists r.X \sqsubseteq^? Y$$

$$\text{local } \mathcal{EL}\text{-unifier } \sigma_1 := \{X \mapsto A, Y \mapsto \top\}$$

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$\rightarrow$ allow also particles of $\sigma(D)$ to occur:

$$\text{local } \mathcal{EL}^{-\top}\text{-unifier } \sigma_2 := \{X \mapsto A \sqcap \exists r.A, Y \mapsto \exists r.A\}$$

Reduction to Linear Language Inclusions

NP reduction to a system of linear language inclusions

$$X_i \subseteq L_0 \cup L_1 X_1 \cup \cdots \cup L_n X_n$$

($L_0, \ldots, L_n$ are subsets of $N_R \cup \{\varepsilon\}$)

A solution θ maps variables to languages over N_R such that

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$$\downarrow$$

$$Y_A \subseteq X_A, \quad X_A \subseteq \{\varepsilon\} \cup Y_A, \quad Y_A \subseteq \{r\}X_A$$

Local Solutions

A solution θ is local if every $w \in \theta(X_A) \setminus \{\varepsilon\}$ occurs on the right-hand side of some inclusion:

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From any finite, local, admissible solution θ we can construct a local $\mathcal{EL}^{-\top}$-unifier of size exponential in $|\Gamma|$ and polynomial in $|\theta|$.

Automata Construction

Finite, local solutions are closed under union.

$\rightarrow$ Check for all X whether there is A and a finite, local solution θ such that $\theta(X_A)$ is non-empty.

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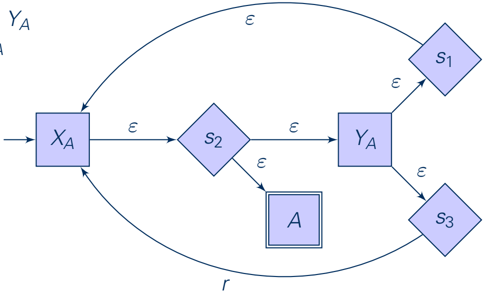
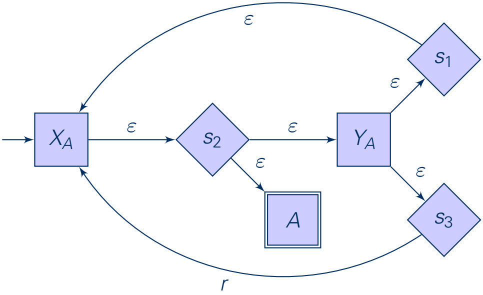
We construct an alternating automaton that accepts the maximal solution for X_A:

$s_1 : \quad Y_A \subseteq X_A$
$s_2 : \quad X_A \subseteq \{\varepsilon\} \cup Y_A$
$s_3 : \quad Y_A \subseteq \{r\} X_A$



The Size of Local $\mathcal{EL}^{-\top}$-Unifiers

Emptiness of this automaton can be checked in PSPACE [Jiang, Ravikumar 1991]. If it is not empty, we can even construct a finite, local solution θ of size at most exponential in $|\Gamma|$:



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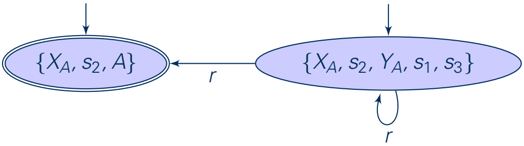

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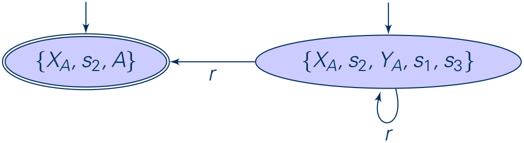
- Construct an equivalent nondeterministic automaton using a powerset construction
- Find a shortest accepting path (of possibly exponential length)
- Extract a local solution (of exponential size) from this path



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Future Work

- $\mathcal{EL}^{(-\top)}$ with general concept inclusion axioms?
- Other concept constructors?
- Implementation of a practical algorithm?

Thank You

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