RECENT ADVANCES IN UNIFICATION FOR THE $\mathcal{EL}$ FAMILY

Franz Baader  Stefan Borgwardt  Barbara Morawska

Manchester, July 1, 2012
The Description Logics $\mathcal{EL}$ and $\mathcal{ELH}_{R+}$

**Syntax**

- concept name
  - $A \in N_C$
- role name
  - $r \in N_R$
The Description Logics $\mathcal{EL}$ and $\mathcal{ELH}_{R^+}$

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<tr>
<td>top concept</td>
<td>$\top \subseteq \Delta^\mathcal{I}$</td>
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<tr>
<td>axioms</td>
<td>$C \sqsubseteq \mathcal{D}$</td>
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<td>$r \circ r \sqsubseteq r^\mathcal{I} \times \Delta^\mathcal{I}$</td>
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Manchester, July 1, 2012 Unification for the $\mathcal{EL}$ Family
The Description Logics $\mathcal{EL}$ and $\mathcal{ELH}_{R^+}$

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subsumption $C \sqsubseteq O$ $D^\mathcal{I} \subseteq D^\mathcal{I}$ (in all models of $O$)

equivalence $C \equiv O$ $D^\mathcal{I} = \Delta^\mathcal{I}$
The Description Logics $\mathcal{EL}$ and $\mathcal{ELH}_{R+}$

**Syntax**

- **concept name** $A \in \mathbb{N}_C$
- **role name** $r \in \mathbb{N}_R$
- **conjunction** $C \sqcap D$
- **existential restriction** $\exists r.C$
- **top concept** $\top$

**interpretation $\mathcal{I} = (\mathcal{I}, \Delta^\mathcal{I})$**

- $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$
- $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$
- $C^\mathcal{I} \cap D^\mathcal{I}$
- $\{x \mid \exists y : (x, y) \in r^\mathcal{I} \land y \in C^\mathcal{I}\}$
- $\Delta^\mathcal{I}$

**axioms**

- **GCI** $C \sqsubseteq D$
- **transitivity axiom** $r \circ r \sqsubseteq r$
- **role hierarchy axiom** $r \sqsubseteq s$

**consequences of an $\mathcal{ELH}_{R+}$-ontology $\mathcal{O}$ (finite set of axioms)**

- **subsumption** $C \sqsubseteq_{\mathcal{O}} D$
- **equivalence** $C \equiv_{\mathcal{O}} D$

$$C^\mathcal{I} \subseteq D^\mathcal{I} \quad \text{(in all models of } \mathcal{O})$$

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**Axioms**

- **GCI**
  - $C \sqsubseteq D$
  - $C^\mathcal{I} \subseteq D^\mathcal{I}$

- **Transitivity axiom**
  - $r \circ r \sqsubseteq r$
  - $r^\mathcal{I} \circ r^\mathcal{I} \subseteq r^\mathcal{I}$

- **Role hierarchy axiom**
  - $r \sqsubseteq s$
  - $r^\mathcal{I} \subseteq s^\mathcal{I}$

**Consequences of an $\mathcal{ELH}_{R+}$-ontology $\mathcal{O}$ (finite set of axioms)**

- **Subsumption**
  - $C \sqsubseteq_\mathcal{O} D$
  - $C^\mathcal{I} \subseteq D^\mathcal{I}$

- **Equivalence**
  - $C \equiv_\mathcal{O} D$
  - $C^\mathcal{I} = D^\mathcal{I}$

Subsumption can be checked in polynomial time. [Baader, Brandt, Lutz IJCAI’05]

If $\mathcal{O}$ contains only GCIs, we call it an $\mathcal{EL}$-ontology.

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Unification in $\mathcal{EL}$

Some concept names are variables ($X \in N_v$), all others are constants ($A \in N_c$).

→ unification problem: $\Gamma = \{C_1 \equiv? D_1, \ldots, C_n \equiv? D_n\}$
   (w.r.t. a ground ontology $\mathcal{O}$)

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A unifier $\sigma$ (w.r.t. $\mathcal{O}$) substitutes variables with concept terms such that

$$\sigma(C_1) \equiv_{\mathcal{O}} \sigma(D_1), \ldots, \sigma(C_n) \equiv_{\mathcal{O}} \sigma(D_n).$$
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$$\sigma(C_1) \equiv_{\mathcal{O}} \sigma(D_1), \ldots, \sigma(C_n) \equiv_{\mathcal{O}} \sigma(D_n).$$

Unification modulo the theory of bounded semilattices with monotone operators:

- $A$ → free constant
- $X$ → variable
- $\sqcap$ → binary associative, commutative, idempotent operator
- $\exists r.$ → unary monotone operator
- $\top$ → constant; unit for $\sqcap$
- $C \sqsubseteq D$ → ground identity
- $r \circ r \sqsubseteq r$ → non-ground identity ($\exists r. \exists r. X \sqsubseteq \exists r. X$)
Why only ground ontologies?

\[ C \equiv O D \text{ iff } t_C = \text{SLmO} \cup G \cup \mathcal{E} \cup E \cup H_{R^+} \quad t_D \]

Manchester, July 1, 2012  Unification for the \( \mathcal{E} \mathcal{L} \) Family
Why only ground ontologies?

\[ C \equiv \emptyset \text{ iff } t_C = \text{SLmO } \cup \text{GO,EL } \cup \text{EO,HR}^{+} t_D \]

\text{o ground: } SLmO-unification with additional identities:

\[ \sigma(s_i) = \text{SLmO } \cup \text{GO } \cup \text{EO } \sigma(t_i) \]
Why only ground ontologies?

\[ C \equiv O \iff t_C = SLmO \cup G \cup E \cup \mathcal{H}_{R^+} t_D \]

\( O \) ground: \( SLmO \)-unification with additional identities:

\[ \sigma(s_i) = SLmO \cup G \cup E \sigma(t_i) \]

\( O \) not ground: Rigid \( G \)-unification with background theory \( SLmO \) and additional identities:

\[ \sigma(s_i) = SLmO \cup \sigma(G) \cup E \sigma(t_i) \]

Simultaneous rigid \( E \)-unification is undecidable. [Degtyarev, Voronkov 1996]
Results

Unification in $\mathcal{EL}$ w.r.t. $\mathcal{O} = \emptyset$ is NP-complete.

- Matching is NP-hard. [Küsters ’01]
- Unification is in NP. [Baader, Morawska RTA’09/LMCS’10/LPAR’10]
- We can restrict the search to local unifiers of polynomial size.

Unification in $\mathcal{EL} - \top$ w.r.t. $\mathcal{O} = \emptyset$ is PACE-complete. [Baader, Binh, Borgwardt, Morawska CADE’11]

- Local unifiers may be of exponential size. [Baader, Binh, Borgwardt, Morawska UNIF’11]

Unification in $\mathcal{EL} w.r.t. \mathcal{ELH}^{R+ontologies}$ is NP-complete.

- Brute-force algorithm for $\mathcal{EL}$-ontologies [KR’12]
- Rule-based algorithm for $\mathcal{EL}$-ontologies [DL ‘12]
- SAT translation for $\mathcal{ELH}^{R+ontologies}$ [IJCAR’12]

- Again, local unifiers are of polynomial size.

Unification in $\mathcal{ALCO}$ and $\mathcal{SHI}$ is undecidable. [Wolter, Zakharyaschev 2008]
Results

Unification in $\mathcal{EL}$ w.r.t. $\mathcal{O} = \emptyset$ is **NP-complete**.
- Matching is NP-hard. [Küsters ’01]
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Unification in $\mathcal{EL}^{-T}$ w.r.t. $\mathcal{O} = \emptyset$ is **PSPACE-complete**.
[Baader, Binh, Borgwardt, Morawska CADE’11]
- Local unifiers may be of exponential size.
[Baader, Binh, Borgwardt, Morawska UNIF’11]
Results

Unification in $\mathcal{E}L$ w.r.t. $\mathcal{O} = \emptyset$ is NP-complete.
- Matching is NP-hard. [Küstes ’01]
- Unification is in NP. [Baader, Morawska RTA’09/LMCS’10/LPAR’10]
- We can restrict the search to local unifiers of polynomial size.

Unification in $\mathcal{EL}^{-T}$ w.r.t. $\mathcal{O} = \emptyset$ is PSPACE-complete.
[Baader, Binh, Borgwardt, Morawska CADE’11]
- Local unifiers may be of exponential size.
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Unification in $\mathcal{EL}$ w.r.t. $\mathcal{ELS}H_{R^+}$-ontologies is NP-complete.
- Brute-force algorithm for $\mathcal{EL}$-ontologies [KR’12]
- Rule-based algorithm for $\mathcal{EL}$-ontologies [DL’12]
- SAT translation for $\mathcal{ELS}H_{R^+}$-ontologies [IJCAR’12]
- Again, local unifiers are of polynomial size.
Results

Unification in $\mathcal{EL}$ w.r.t. $\mathcal{O} = \emptyset$ is **NP-complete**.
- Matching is NP-hard. [Küsters ’01]
- Unification is in NP. [Baader, Morawska RTA’09/LMCS’10/LPAR’10]
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Unification in $\mathcal{EL}^{-\top}$ w.r.t. $\mathcal{O} = \emptyset$ is **PSPACE-complete**.
[Baader, Binh, Borgwardt, Morawska CADE’11]
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Unification in $\mathcal{EL}$ w.r.t. $\mathcal{ELH}_{R+}$-ontologies is **NP-complete**.
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Unification in $\mathcal{ALCO}$ and $\mathcal{SHI}$ is **undecidable**. [Wolter, Zakharyaschev 2008]
Characterization of Subsumption for Ground Concepts

for $\mathcal{O} = \emptyset$:

- $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq_{\emptyset} D_1 \sqcap \cdots \sqcap D_m$ iff for all $D_j$ there is a $C_i$ with $C_i \sqsubseteq_{\emptyset} D_j$. 
Characterization of Subsumption for Ground Concepts

for $\mathcal{O} = \emptyset$:

- $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq_{\emptyset} D_1 \sqcap \cdots \sqcap D_m$ iff for all $D_j$ there is a $C_i$ with $C_i \sqsubseteq_{\emptyset} D_j$.
- $\exists r. C \sqsubseteq_{\emptyset} \exists s. D$ iff $r = s$ and $C \sqsubseteq_{\emptyset} D$. 

...for $\mathcal{O} \neq \emptyset$:

- more cases for axioms in $\mathcal{O}$
- $A \sqsubseteq \mathcal{O} B$ with $A, B \in \mathcal{N}$ can hold, e.g., if $A \sqsubseteq \exists r. C$, $C \sqsubseteq D$, $\exists s. D \sqsubseteq B$, $r \sqsubseteq s$ are in $\mathcal{O}$.

- two different characterizations in [KR’12] (for EL) and [IJCAR’12] (for ELH+).
Characterization of Subsumption for Ground Concepts

for $\mathcal{O} = \emptyset$:

- $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq_\emptyset D_1 \sqcap \cdots \sqcap D_m$ iff for all $D_j$ there is a $C_i$ with $C_i \sqsubseteq_\emptyset D_j$.
- $\exists r. C \sqsubseteq_\emptyset \exists s. D$ iff $r = s$ and $C \sqsubseteq_\emptyset D$.
- $A \sqsubseteq_\emptyset B$ for $A, B \in N_C$ iff $A = B$. 


Characterization of Subsumption for Ground Concepts

for $\mathcal{O} = \emptyset$:  
- $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq \emptyset D_1 \sqcap \cdots \sqcap D_m$ iff for all $D_j$ there is a $C_i$ with $C_i \sqsubseteq \emptyset D_j$.  
- $\exists r. C \sqsubseteq \emptyset \exists s. D$ iff $r = s$ and $C \sqsubseteq \emptyset D$.  
- $A \sqsubseteq \emptyset B$ for $A, B \in \mathbb{N}_C$ iff $A = B$.  
- $A \not\sqsubseteq \emptyset \exists r. B, \exists r. B \not\sqsubseteq \emptyset A, \ldots$
Characterization of Subsumption for Ground Concepts

for $\mathcal{O} = \emptyset$:

- $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq_{\emptyset} D_1 \sqcap \cdots \sqcap D_m$ iff for all $D_j$ there is a $C_i$ with $C_i \sqsubseteq_{\emptyset} D_j$.
- $\exists r. C \sqsubseteq_{\emptyset} \exists s. D$ iff $r = s$ and $C \sqsubseteq_{\emptyset} D$.
- $A \sqsubseteq_{\emptyset} B$ for $A, B \in \mathbb{N}_C$ iff $A = B$.
- $A \not\sqsubseteq_{\emptyset} \exists r. B$, $\exists r. B \not\sqsubseteq_{\emptyset} A$, ...

for $\mathcal{O} \neq \emptyset$:

- more cases for axioms in $\mathcal{O}$
- $A \sqsubseteq_{\mathcal{O}} B$ with $A, B \in \mathbb{N}_C$ and $A \neq B$ can hold, e.g., if

$$A \sqsubseteq \exists r. C, \ C \sqsubseteq D, \ \exists s. D \sqsubseteq B, \ r \sqsubseteq s$$

are in $\mathcal{O}$. 

Manchester, July 1, 2012  
Unification for the $\mathcal{EL}$ Family
Characterization of Subsumption for Ground Concepts

for $\mathcal{O} = \emptyset$:

- $C_1 \sqcap \cdots \sqcap C_n \subseteq_\emptyset D_1 \sqcap \cdots \sqcap D_m$ iff for all $D_j$ there is a $C_i$ with $C_i \subseteq_\emptyset D_j$.
- $\exists r. C \subseteq_\emptyset \exists s. D$ iff $r = s$ and $C \subseteq_\emptyset D$.
- $A \subseteq_\emptyset B$ for $A, B \in \mathcal{N}_C$ iff $A = B$.
- $A \not\subseteq_\emptyset \exists r. B, \exists r. B \not\subseteq_\emptyset A, \ldots$

for $\mathcal{O} \neq \emptyset$:

- more cases for axioms in $\mathcal{O}$
- $A \subseteq_\mathcal{O} B$ with $A, B \in \mathcal{N}_C$ and $A \neq B$ can hold, e.g., if

$$A \subseteq \exists r. C, \ C \subseteq D, \ \exists s. D \subseteq B, \ r \subseteq s$$

are in $\mathcal{O}$.
- two different characterizations in [KR’12] (for $\mathcal{EL}$) and [IJCAR’12] (for $\mathcal{ELH}_{R+}$)
Locality and the Brute-Force Approach

**flat atom**: concept name or existential restriction $\exists r. A$ with $A \in N_C$

**assumption**: unification problem $\Gamma$ and all GCIs in the $\mathcal{ELH}_{R^+}$-ontology $\mathcal{O}$ contain only (conjunctions of) flat atoms
Locality and the Brute-Force Approach

flat atom: concept name or existential restriction \( \exists r.A \) with \( A \in N_C \)

assumption: unification problem \( \Gamma \) and all GCIs in the \( \mathcal{ELH}_{R+} \)-ontology \( \mathcal{O} \) contain only (conjunctions of) flat atoms

\( \text{At}_{tr} := \) atoms in the unification problem and \( \mathcal{O} \) ("closed under transitive roles")

\( \text{At}_{nv} := \text{At}_{tr} \setminus N_v \)
Locality and the Brute-Force Approach

flat atom: concept name or existential restriction $\exists r.A$ with $A \in N_C$

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$A_{tr} :=$ atoms in the unification problem and $\mathcal{O}$ (“closed under transitive roles”)
$A_{nv} := A_{tr} \setminus N_v$

acyclic assignment $S$:

$S_X = \{B\} \quad \sigma_S(X) = B$
$S_Y = \{\exists s.A, \exists r.X\} \quad \sigma_S(Y) = \exists s.A \land \exists r.B$

↑
local substitution
Locality and the Brute-Force Approach

flat atom: concept name or existential restriction $\exists r. A$ with $A \in N_C$

assumption: unification problem $\Gamma$ and all GCIs in the $\mathcal{ELH}_{R^+}$-ontology $O$ contain only (conjunctions of) flat atoms

$At_{tr} :=$ atoms in the unification problem and $O$ (“closed under transitive roles”)
$At_{nv} := At_{tr} \setminus N_v$

acyclic assignment $S$:

$S_X = \{B, \exists r. X\}$
$S_Y = \{\exists s. A, \exists r. X\}$

$\sigma_S(X) = B \cap ?$
$\sigma_S(Y) = \exists s. A \cap \exists r. (B \cap ?)$

↑
local substitution
Locality and the Brute-Force Approach

**flat atom**: concept name or existential restriction $\exists r. A$ with $A \in NC$

**assumption**: unification problem $\Gamma$ and all GCIs in the $\mathcal{ELH}_{R^+}$-ontology $\mathcal{O}$ contain only (conjunctions of) flat atoms

$At_{tr} :=$ atoms in the unification problem and $\mathcal{O}$ ("closed under transitive roles")

$At_{nv} := At_{tr} \setminus N_v$

**acyclic assignment $S$**:

$S_X = \{ B \}$  \hspace{5cm} $\sigma_S(X) = B$

$S_Y = \{ \exists s. A, \exists r. X \}$  \hspace{5cm} $\sigma_S(Y) = \exists s. A \sqcap \exists r. B$

↑

local substitution

**Goal**: Every unifiable unification problem has a local unifier.
Necessary Restriction

\( \mathcal{O} \) is cycle-restricted if

\[ C \not\sqsubseteq_{\mathcal{O}} \exists r_1 \ldots \exists r_n. C \]

holds for all concept descriptions \( C \) and \( r_1, \ldots, r_n \in N_R, n \geq 1. \)
Necessary Restriction

\( \mathcal{O} \) is cycle-restricted if

\[
C \not\subseteq_{\mathcal{O}} \exists r_1 \ldots \exists r_n.C
\]

holds for all concept descriptions \( C \) and \( r_1, \ldots, r_n \in N_R, n \geq 1 \).

This ensures that variables have no cyclic dependencies under unifiers, i.e., that there is no subsumption \( \gamma(X) \subseteq_{\mathcal{O}} \exists r_1 \ldots \exists r_n.\gamma(X) \) for any substitution \( \gamma \).

If \( \mathcal{O} \) is cycle-restricted, then every unifiable unification problem has a local unifier.

[KR’12, IJCAR’12]
Necessary Restriction

\[ \mathcal{O} \text{ is cycle-restricted if} \]
\[ C \nsubseteq_\mathcal{O} \exists r_1 \ldots \exists r_n. C \]
holds for all concept descriptions \( C \) and \( r_1, \ldots, r_n \in \mathbb{N}_R, n \geq 1 \).

This ensures that variables have no cyclic dependencies under unifiers, i.e., that there is no subsumption \( \gamma(X) \nsubseteq_\mathcal{O} \exists r_1 \ldots \exists r_n. \gamma(X) \) for any substitution \( \gamma \).

If \( \mathcal{O} \) is cycle-restricted, then every unifiable unification problem has a local unifier. \[ \text{[KR’12, IJCAR’12]} \]

Cycle-restrictedness can be checked in polynomial time.
Rule-based approach

nondeterministic rules can be applied to a subsumption in order to solve it:

\[ \exists r.X \sqcap \exists r.Y \sqsubseteq \exists r.A \]

with more GCIs in \( O \), we have more choices:

\[ C \sqsubseteq O \exists r.B \]

\[ B \sqsubseteq \exists r.X \text{ (ground)} \quad \text{and} \quad C \sqsubseteq \exists r.X \text{ (ground)} \]

\[ X \sqsubseteq A \]

\[ Y \sqsubseteq A \]
Rule-based approach

nondeterministic rules can be applied to a subsumption in order to solve it:

\[ \exists r.X \sqsubseteq \exists r.Y \sqsubseteq \? \exists r.A \]

- \[ X \sqsubseteq \? A \rightarrow S_X := S_X \cup \{A\} \]
- \[ Y \sqsubseteq \? A \rightarrow S_Y := S_Y \cup \{A\} \]
Rule-based approach

nondeterministic rules can be applied to a subsumption in order to solve it:

\[ \exists r. X \sqsubseteq \exists r. Y \sqsubseteq \exists r. A \]

\[ Y \sqsubseteq A \rightarrow S_y := S_y \cup \{A\} \]

eager rules are always applied first:

\[ \exists r. A \sqsubseteq B \quad \text{(ground)} \quad \rightarrow \quad \text{fail if } \exists r. A \not\sqsubseteq \mathcal{O} B \]
Rule-based approach

nondeterministic rules can be applied to a subsumption in order to solve it:

\[ \exists r. X \sqsubseteq^? A \rightarrow S_X := S_X \cup \{A\} \]

\[ \exists r. Y \sqsubseteq^? A \rightarrow S_Y := S_Y \cup \{A\} \]

eager rules are always applied first:

\[ \exists r. A \sqsubseteq^? B \]

(ground)

\[ \text{fail if } \exists r. A \not\sqsubseteq^? B \]

with more GCIs in \( \mathcal{O} \), we have more choices:

\[ C \sqsubseteq^? \exists r. X \]

(C ground)

\[ C \sqsubseteq^? \exists r. B \rightarrow B \sqsubseteq^? X \]
SAT translation

propositional variables \([C \sqsubseteq D]\) for all \(C, D \in \text{At}_{\text{tr}}\)

valuation \(\tau \rightarrow S_X\) contains \(D\) iff \(\tau([X \sqsubseteq D]) = 1\)
SAT translation

propositional variables \([C \sqsubseteq D]\) for all \(C, D \in \text{At}_{\text{tr}}\)

valuation \(\tau \rightarrow S_X\) contains \(D\) iff \(\tau([X \sqsubseteq D]) = 1\)

\[\sigma_S(X) \subseteq \sigma_S(D)\]
SAT translation

Propositional variables \([C \sqsubseteq D]\) for all \(C, D \in \text{At}_{\text{tr}}\)

valuation \(\tau \rightarrow S_X\) contains \(D\) iff \(\tau([X \sqsubseteq D]) = 1\)

\[\sigma_S(X) \sqsubseteq \sigma_S(D)\]

Propositional formulae:

\[\exists r.A \sqsubseteq \exists r.X \rightarrow [A \sqsubseteq X]\]
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+ clauses asserting the subsumptions in the unification problem
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+ clauses forcing acyclicity of the induced assignment \(S\)
Minimal Unifiers

Both algorithms express the unification problem using the characterization of subsumption.

**Soundness:** acyclicity of $S$, soundness of the characterization

**Completeness:** cycle-restrictedness of $O$, completeness of the characterization
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These algorithms cannot be modified to yield exactly the $\succ_\mathcal{X}$-minimal unifiers while staying in NP. [AiML’12]
Conclusions

Summary:

• three NP-algorithms for $\mathcal{ELH}_{R^+}$-unification w.r.t. cycle-restricted ontologies
• $\succ_{N_v}$-minimal unifiers are enough, but more difficult to find
• implementation for empty ontologies

Future Work:

• extend the implementation to cycle-restricted $\mathcal{ELH}_{R^+}$-ontologies
• general ontologies? fixpoint semantics for cyclic assignments?
• extensions of $\mathcal{ELH}_{R^+}$ ($\mathcal{EL}^+$$,$ $\mathcal{FLE}$$,$ $\mathcal{ALC}$)?
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