FUZZY DESCRIPTION LOGICS WITH GENERAL CONCEPT INCLUSIONS

Verteidigung der Dissertation

Stefan Borgwardt

Dresden, 23.05.2014
Outline

• Introduction to Fuzzy Description Logics
• Fuzzy Description Logics over \([0, 1]\)
• Fuzzy Description Logics over Finite Lattices
• Summary
Motivation

laura: Human
laura: Female
laura: Happy
(laura, 2): has-age
laura: ∃ sits-on. Swing
(laura, elisabeth): has-sister
Human ∩ ∃ has-age.(≤12) ⊆ Child

elisabeth: Human
elisabeth: Female
elisabeth: Happy
(elisabeth, 4): has-age
elisabeth: ∃ wears.Hood
(laura, elisabeth): likes
Child ∩ Female ⊆ Girl
Motivation

laura: Human
laura: Female
laura: Happy
(laura, 2): has-age
laura: ∃ sits-on.Swing
(laura, elisabeth): has-sister

elisabeth: Human
elisabeth: Female
elisabeth: Happy
(elisabeth, 4): has-age
elisabeth: ∃ wears.Hood
(laura, elisabeth): likes

Human ⊓ ∃ has-age.(≤12) ⊑ Child
Child ⊓ Female ⊑ Girl

... holds to degree / probability / possibility 0 ... 0.1 ... 0.5 ... 0.9 ... 1
Uncertainty and Vagueness

\[ \text{laura: } \exists \text{sits-on. Swing} \]
Uncertainty and Vagueness

\( \text{laura} : \exists \text{sits-on.Swing} \)

Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds
Uncertainty and Vagueness

\[ \text{laura}: \exists \text{sits-on}.Swing \]

Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds
- \( P\text{-SHOIN(D)} \) (Lukasiewicz 2008)
- Prob-\(\text{ALC} \) (Lutz and Schröder 2010)
- \( \text{ALCN} \) with possibilistic axioms (Hollunder 1995)

Dresden, 23.05.2014

Fuzzy Description Logics with GCIs
Uncertainty and Vagueness

\textit{laura}: \exists \text{sits-on}.\text{Swing}

Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds
- \textit{P-SHOIN(D)} \quad \text{(Lukasiewicz 2008)}
- \text{Prob-ALC} \quad \text{(Lutz and Schröder 2010)}
- \textit{ALCN} with possibilistic axioms \quad \text{(Hollunder 1995)}

\textit{laura}: \text{Happy}, \ (\text{laura, elisabeth}) : \text{likes}
Uncertainty and Vagueness

\[ \text{laura: } \exists \text{sits-on.Swing} \]

**Probabilistic/Possibilistic DLs:**

- probability/possibility distributions on possible worlds
- \( PSHOIN(D) \) (Lukasiewicz 2008)
- Prob-\( ALC \) (Lutz and Schröder 2010)
- \( ALCN \) with possibilistic axioms (Hollunder 1995)

\[ \text{laura: Happy, (laura, elisabeth): likes} \]

**Fuzzy DLs:** (Straccia 1998; Tresp and Molitor 1998; Yen 1991)

- two degrees of truth (\textit{false}, \textit{true}) are replaced by \([0, 1]\) (Zadeh 1965)
Uncertainty and Vagueness

Laura: \( \exists \text{sits-on}.\text{Swing} \)

Probabilistic/Possibilistic DLs:
- probability/possibility distributions on possible worlds
- \( P\text{-SHOIN}(D) \) (Lukasiewicz 2008)
- Prob-\( ALC \) (Lutz and Schröder 2010)
- \( ALCN \) with possibilistic axioms (Hollunder 1995)

Laura: Happy, (Laura, Elisabeth): likes

Fuzzy DLs: (Straccia 1998; Tresp and Molitor 1998; Yen 1991)
- two degrees of truth (false, true) are replaced by \([0, 1]\) (Zadeh 1965)
- statements are assigned a truth degree
- conjunction, etc. are interpreted by appropriate truth functions
Mathematical Fuzzy Logic

\[ \text{laura: Happy} \sqcap \text{Cute} \]
Mathematical Fuzzy Logic

\[ \text{laura}: \text{Happy} \sqcap \text{Cute} \]

- t-norm \( \otimes: [0, 1] \times [0, 1] \rightarrow [0, 1] \):
  associative, commutative, monotone, unit 1, (continuous)

(Hájek 2001)
Mathematical Fuzzy Logic

\textit{laura}: Happy \sqcap Cute

- \textit{t-norm} $\otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]$: associative, commutative, monotone, unit 1, (continuous) \cite{Hajek2001}

- residuum $\Rightarrow: [0, 1] \times [0, 1] \rightarrow [0, 1]$: $x \otimes y \leq z \iff y \leq (x \Rightarrow z)$

- residual negation $\ominus x := x \Rightarrow 0$
Mathematical Fuzzy Logic

\textit{laura}: \textit{Happy} \sqcap \textit{Cute}

- **t-norm** $\otimes: [0, 1] \times [0, 1] \to [0, 1]$: associative, commutative, monotone, unit 1, (continuous) 

- **residuum** $\Rightarrow: [0, 1] \times [0, 1] \to [0, 1]$: $x \otimes y \leq z$ iff $y \leq (x \Rightarrow z)$

- **residual negation** $\ominus x := x \Rightarrow 0$

- **involutive negation** $\sim x := 1 - x$

- **t-conorm** $x \oplus y := \sim(\sim x \otimes \sim y)$

(Hájek 2001)
Mathematical Fuzzy Logic

laura: Happy $\sqcap$ Cute

- t-norm $\otimes$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$: associative, commutative, monotone, unit 1, (continuous) (Hájek 2001)

Continuous t-norms:
- Gödel (G): $x \otimes y = \min\{x, y\}$

Dresden, 23.05.2014 Fuzzy Description Logics with GCIs
Mathematical Fuzzy Logic

\[ \text{laura: Happy } \sqcap \text{ Cute} \]

- **t-norm** \( \otimes : [0, 1] \times [0, 1] \to [0, 1] \):
  - associative, commutative, monotone, unit 1, (continuous)

Continuous t-norms:
- **Gödel** (\( G \)): \( x \otimes y = \min\{x, y\} \)
- **Product** (\( \Pi \)): \( x \otimes y = x \cdot y \)
laura: Happy ⊓ Cute

- $t$-norm $\otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]$: associative, commutative, monotone, unit 1, (continuous)  
  (Hájek 2001)

Continuous $t$-norms:
- Gödel ($G$): $x \otimes y = \min\{x, y\}$
- Product ($\Pi$): $x \otimes y = x \cdot y$
- Łukasiewicz ($Ł$): $x \otimes y = \max(0, x + y - 1)$
Mathematical Fuzzy Logic

Laura: Happy \( \cap \) Cute

- **t-norm** \( \otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]: \)
  associative, commutative, monotone, unit 1, (continuous)

Continuous t-norms:
- Gödel \( (G) \): \( x \otimes y = \min\{x, y\} \)
- Product \( (\Pi) \): \( x \otimes y = x \cdot y \)
- Łukasiewicz \( (L) \): \( x \otimes y = \max(0, x + y - 1) \)
- Ordinal sums
Mathematical Fuzzy Logic

- **t-norm** $\otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]$: associative, commutative, monotone, unit 1, (continuous) 

(Hájek 2001)

Continuous t-norms:

- Gödel ($G$): $x \otimes y = \min\{x, y\}$
- Product ($\Pi$): $x \otimes y = x \cdot y$
- Łukasiewicz ($\mathcal{L}$): $x \otimes y = \max(0, x + y - 1)$
- Ordinal sums

Dresden, 23.05.2014 Fuzzy Description Logics with GCIs
Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: $\text{Happy}^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]$
- role names: $\text{likes}^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]$
- individual names: $\text{laura}^\mathcal{I} \in \Delta^\mathcal{I}$
Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:
- concept names: $\text{Happy}^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]$  
- role names: $\text{likes}^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]$  
- individual names: $\text{laura}^\mathcal{I} \in \Delta^\mathcal{I}$

$\mathcal{FL}$:
- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$  
- top $\top^\mathcal{I}(x) = 1$  
- existential restriction $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$

$\mathcal{EL}$:
- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$  
- top $\top^\mathcal{I}(x) = 1$  
- existential restriction $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$
Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- concept names: $\text{Happy}^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]$
- role names: $\text{likes}^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]$
- individual names: $\text{laura}^\mathcal{I} \in \Delta^\mathcal{I}$

$\mathcal{EL}$:

- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$
- top $\top^\mathcal{I}(x) = 1$
- existential restriction $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$

Witnessed interpretations: $(\exists r.C)^\mathcal{I}(x) = \max_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$ (Hájek 2005)
Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:
- concept names: $\text{Happy}^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]$
- role names: $\text{likes}^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]$
- individual names: $\text{laura}^\mathcal{I} \in \Delta^\mathcal{I}$

$\mathcal{EL}$:
- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$
- top $\top^\mathcal{I}(x) = 1$
- existential restriction $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$

Witnessed interpretations: $(\exists r.C)^\mathcal{I}(x) = \max_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$ (Hájek 2005)

More constructors:
- $\neg C$ : involutive negation
- $\Box C$ : residual negation
- $\Rightarrow C \rightarrow D$ : implication
- $\bot$ : bottom
Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^\mathcal{I}, .^\mathcal{I})$:
- concept names: $\text{Happy}^\mathcal{I} : \Delta^\mathcal{I} \to [0, 1]$
- role names: $\text{likes}^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0, 1]$
- individual names: $\text{laura}^\mathcal{I} \in \Delta^\mathcal{I}$

$\mathcal{EL}$:
- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$
- top $\top^\mathcal{I}(x) = 1$
- existential restriction $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$

Witnessed interpretations: $(\exists r.C)^\mathcal{I}(x) = \max_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$ (Hájek 2005)

More constructors:
- $\neg$ : involutive negation $\neg C$
- $\Box$ : residual negation $\Box C$
- $\Rightarrow$ : implication $C \Rightarrow D$, bottom $\bot$

\[\otimes-\text{IELC} \quad \Pi-\text{NELC} \quad \text{L-ELC} \quad \otimes-\text{ISCHOT}\]

fuzzy extensions of $\text{ALC}$

Dresden, 23.05.2014 Fuzzy Description Logics with GCI
Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a: C \triangleright p \rangle$: $C^I(a^I) \triangleright p$
- role assertion $\langle (a, b): r \triangleright p \rangle$: $r^I(a^I, b^I) \triangleright p$
- GCI $\langle C \sqsubseteq D \triangleright p \rangle$: $C^I(x) \Rightarrow D^I(x) \geq p$ for all $x \in \Delta^I$
Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a : C \triangleright p \rangle$: $C^\mathcal{I}(a^\mathcal{I}) \triangleright p$
- role assertion $\langle (a, b) : r \triangleright p \rangle$: $r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \triangleright p$
- GCI $\langle C \sqsubseteq D \triangleright p \rangle$: $C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geq p$ for all $x \in \Delta^\mathcal{I}$
  
  $$p \otimes C^\mathcal{I}(x) \leq D^\mathcal{I}(x)$$

Reasoning tasks:

- ontology consistency: Does $\mathcal{O}$ have a (witnessed) model?
- concept satisfiability: Is there a (witnessed) model $\mathcal{I}$ of $\mathcal{O}$ with $C^\mathcal{I}(x) \geq p$ for some $x \in \Delta^\mathcal{I}$?

Applications: (Ciaramella et al. 2010; Meghini, Sebastiani, and Straccia 2001)

- recommender systems with background knowledge
- information retrieval, query relaxation

Dresden, 23.05.2014  Fuzzy Description Logics with GCIs
Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a: C \triangleright p \rangle$: $C^\mathcal{I}(a^\mathcal{I}) \triangleright p$
- role assertion $\langle (a, b): r \triangleright p \rangle$: $r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \triangleright p$
- GCI $\langle C \sqsubseteq D \geq p \rangle$: $C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geq p$ for all $x \in \Delta^\mathcal{I}$

\[ p \otimes C^\mathcal{I}(x) \leq D^\mathcal{I}(x) \]


Applications: (Ciaramella et al. 2010; Meghini, Sebastiani, and Straccia 2001)

- recommender systems with background knowledge
- information retrieval, query relaxation
Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a: C \nless p \rangle: C^\mathcal{I}(a^\mathcal{I}) \nless p$
- role assertion $\langle (a, b): r \nless p \rangle: r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \nless p$
- GCI $\langle C \subseteq D \n SL p \rangle$: $C^\mathcal{I}(x) \implies D^\mathcal{I}(x) \geq p$ for all $x \in \Delta^\mathcal{I}$

\[
p \otimes C^\mathcal{I}(x) \leq D^\mathcal{I}(x)
\]

Reasoning tasks:

- ontology consistency: Does $\mathcal{O}$ have a (witnessed) model?
- concept satisfiability: Is there a (witnessed) model $\mathcal{I}$ of $\mathcal{O}$ with $C^\mathcal{I}(x) \geq p$ for some $x \in \Delta^\mathcal{I}$?
Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:
- concept assertion $\langle a: C \triangleright p \rangle$: $C^\mathcal{I}(a^\mathcal{I}) \triangleright p$
- role assertion $\langle (a, b): r \triangleright p \rangle$: $r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \triangleright p$
- GCI $\langle C \sqsubseteq D \geq p \rangle$: $C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geq p$ for all $x \in \Delta^\mathcal{I}$
  \[ p \otimes C^\mathcal{I}(x) \leq D^\mathcal{I}(x) \]

Reasoning tasks:
- ontology consistency: Does $\mathcal{O}$ have a (witnessed) model?
- concept satisfiability: Is there a (witnessed) model $\mathcal{I}$ of $\mathcal{O}$ with $C^\mathcal{I}(x) \geq p$ for some $x \in \Delta^\mathcal{I}$?

Applications: (Ciaramella et al. 2010; Meghini, Sebastiani, and Straccia 2001)
- recommender systems with background knowledge
- information retrieval, query relaxation
Outline

• Introduction to Fuzzy Description Logics
• Fuzzy Description Logics over $[0, 1]$
• Fuzzy Description Logics over Finite Lattices
• Summary
Tableau Algorithms

Tableau algorithm for $\otimes$-$\mathcal{ELC}$ without GCIs: (Bobillo and Straccia 2009)

\[ \langle a: C \geq p \rangle \leadsto \langle a: C = v_a: C \rangle, \ v_a: C \geq p \]
\[ \langle (a, b): r \geq p \rangle \leadsto \langle (a, b): r = v_{(a,b)}: r \rangle, \ v_{(a,b)}: r \geq p \]
Tableau Algorithms

Tableau algorithm for $\otimes{\mathcal{ELC}}$ without GCIs: (Bobillo and Straccia 2009)

\[
\langle a: C \geq p \rangle \leadsto \langle a: C = v_a:C \rangle, v_a:C \geq p
\]

\[
\langle (a, b): r \geq p \rangle \leadsto \langle (a, b): r = v_{(a,b):r} \rangle, v_{(a,b):r} \geq p
\]

\[
\langle x: C \cap D = v \rangle \leadsto \langle x: C = v_x:C \rangle, \langle x: D = v_x:D \rangle, v = v_x:C \otimes v_x:D
\]

...
Tableau Algorithms

Tableau algorithm for $\otimes$-$\mathcal{ELC}$ without GCIs: (Bobillo and Straccia 2009)

$$\langle a: C \geq p \rangle \rightsquigarrow \langle a: C = v_{a:C}, v_{a:C} \geq p \rangle$$

$$\langle (a, b): r \geq p \rangle \rightsquigarrow \langle (a, b): r = v_{(a, b):r}, v_{(a, b):r} \geq p \rangle$$

$$\langle x: C \sqcap D = v \rangle \rightsquigarrow \langle x: C = v_{x:C}, x: D = v_{x:D}, v = v_{x:C} \otimes v_{x:D} \rangle$$

$$\ldots$$

$$\langle x: \exists r. C = v \rangle \rightsquigarrow \langle (x, y): r = v_{(x, y):r}, y: C = v_{y:C}, v = v_{(x, y):r} \otimes v_{y:C} \rangle$$

$$\langle x: \exists r. C = v \rangle, \langle (x, y): r = v' \rangle \rightsquigarrow \langle y: C = v_{y:C}, v' \geq v' \otimes v_{y:C} \rangle$$
Tableau Algorithms

Tableau algorithm for $\otimes$-\textit{ELC} without GCIs: \cite{Bobillo and Straccia 2009}

\[
\langle a: C \geq p \rangle \rightsquigarrow \langle a: C = v_{a:C}, \ v_{a:C} \geq p \rangle \\
\langle (a, b): r \geq p \rangle \rightsquigarrow \langle (a, b): r = v_{(a,b):r}, \ v_{(a,b):r} \geq p \rangle \\
\langle x: C \sqcap D = v \rangle \rightsquigarrow \langle x: C = v_{x:C}, \ x: D = v_{x:D}, \ v = v_{x:C} \otimes v_{x:D} \rangle \\
\ldots
\]

\[
\langle x: \exists r.C = v \rangle \rightsquigarrow \langle (x, y): r = v_{(x,y):r}, \ y: C = v_{y:C}, \ v = v_{(x,y):r} \otimes v_{y:C} \rangle \\
\langle x: \exists r.C = v \rangle, \langle (x, y): r = v' \rangle \rightsquigarrow \langle y: C = v_{y:C}, \ v \geq v' \otimes v_{y:C} \rangle
\]

- deterministic exponential time
- $\mathcal{O}$ is consistent iff the constraints have a solution (NP-hard)
Tableau Algorithms

Tableau algorithm for $\otimes$-$\mathcal{ELC}$ without GCIs: (Bobillo and Straccia 2009)

\[
\langle a: C \geq p \rangle \rightsquigarrow \langle a: C = v_{a:C} \rangle, \ v_{a:C} \geq p
\]
\[
\langle (a, b): r \geq p \rangle \rightsquigarrow \langle (a, b): r = v_{(a,b):r} \rangle, \ v_{(a,b):r} \geq p
\]
\[
\langle x: C \sqcap D = v \rangle \rightsquigarrow \langle x: C = v_{x:C} \rangle, \langle x: D = v_{x:D} \rangle, \ v = v_{x:C} \otimes v_{x:D}
\]
\[\ldots\]
\[
\langle x: \exists r. C = v \rangle \rightsquigarrow \langle (x, y): r = v_{(x,y):r} \rangle, \langle y: C = v_{y:C} \rangle, \ v = v_{(x,y):r} \otimes v_{y:C}
\]
\[
\langle x: \exists r. C = v \rangle, \langle (x, y): r = v' \rangle \rightsquigarrow \langle y: C = v_{y:C} \rangle, \ v \geq v' \otimes v_{y:C}
\]

- deterministic exponential time
- $\mathcal{O}$ is consistent iff the constraints have a solution (NP-hard)
- $\text{NEXPTime}$ for Ł-$\mathcal{ELC}$, $\text{EXPSPACE}$ for $\Pi$-$\mathcal{ELC}$
- possible for any finite ordinal sum
Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- **blocking condition** for $\Pi$-ELC with GCIs: (Bobillo and Straccia 2007)
  $x$ is blocked by $y$ if their assertions and constraints are isomorphic
Fuzzy GCIs

- GCIs like $\langle \top \subseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- **blocking condition** for $\Pi$-ELC with GCIs: $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound

(Bobillo and Straccia 2007)

(Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)
Fuzzy GCIs

- GCIs like $\langle \top \subseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- **blocking condition** for $\Pi$-ELC with GCIs: (Bobillo and Straccia 2007) $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)

\[
\langle a: A \geq 0.5 \rangle, \langle A \subseteq \exists r. A \geq 1 \rangle, \langle \top \subseteq \neg \exists r. \top \geq 0.1 \rangle
\]
Fuzzy GCIs

- GCIs like $\langle T \subseteq \exists r. T \geq 1 \rangle$ can lead to cycles in the tableau
- **blocking condition** for $\Pi\text{-}ELC$ with GCIs: 
  x is blocked by y if their assertions and constraints are isomorphic
- the algorithm is not sound

$\langle a : A \geq 0.5 \rangle, \langle A \subseteq \exists r. A \geq 1 \rangle, \langle T \subseteq \neg \exists r. T \geq 0.1 \rangle$

- $0.5 \leq v_a : A$, $v_a : A \leq v_{x_1} : A \cdot v_{(a,x_1)} : r$, $v_{(a,x_1)} : r \leq 0.9$, $v_a : A \Rightarrow v_{x_1} : A$
Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- blocking condition for $\Pi$-ELC with GCIs: (Bobillo and Straccia 2007)
  $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound  
  (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)

\[
\begin{align*}
\langle a : A \geq 0.5 \rangle, & \quad \langle A \sqsubseteq \exists r. A \geq 1 \rangle, & \quad \langle \top \sqsubseteq \neg \exists r. \top \geq 0.1 \rangle \\
0.5 \leq v_{a : A}, & \quad v_{a : A} \leq v_{x_1 : A} \cdot v_{(a,x_1) : r}, & \quad v_{(a,x_1) : r} \leq 0.9, \\
v_{x_1 : A} \leq v_{x_2 : A} \cdot v_{(x_1,x_2) : r}, & \quad v_{(x_1,x_2) : r} \leq 0.9, 
\end{align*}
\]
Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- **blocking condition** for $\Pi$-ELC with GCIs: (Bobillo and Straccia 2007)
  $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound  
  (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)

\[
\langle a : A \geq 0.5 \rangle, \; \langle A \sqsubseteq \exists r.A \geq 1 \rangle, \; \langle \top \sqsubseteq \neg \exists r. \top \geq 0.1 \rangle
\]

\[
0.5 \leq v_{a:A}, \; v_{a:A} \leq v_{x_1:A} \cdot v_{(a,x_1) : r}, \; v_{(a,x_1) : r} \leq 0.9,
\]

\[
v_{x_1:A} \leq v_{x_2:A} \cdot v_{(x_1,x_2) : r}, \; v_{(x_1,x_2) : r} \leq 0.9,
\]

\[
v_{x_2:A} \leq v_{x_3:A} \cdot v_{(x_2,x_3) : r}, \; v_{(x_2,x_3) : r} \leq 0.9
\]
Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- blocking condition for $\Pi$-ELC with GCIs: $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound

$$\langle a: A \geq 0.5 \rangle, \langle A \sqsubseteq \exists r. A \geq 1 \rangle, \langle \top \sqsubseteq \neg \exists r. \top \geq 0.1 \rangle$$

$$0.5 \leq v_{a: A}, \quad v_{a: A} \leq v_{x_1: A} \cdot v_{(a,x_1): r}, \quad v_{(a,x_1): r} \leq 0.9,$$

$$v_{x_1: A} \leq v_{x_2: A} \cdot v_{(x_1,x_2): r}, \quad v_{(x_1,x_2): r} \leq 0.9,$$

$$v_{x_2: A} \leq v_{x_3: A} \cdot v_{(x_2,x_3): r}, \quad v_{(x_2,x_3): r} \leq 0.9$$

$$0.5 \cdot \left( \frac{10}{9} \right)^i \leq v_{x_i: A} \quad \leadsto \quad v_{x_7: A} > 1 !$$
Fuzzy GCIs

- GCIs like $\langle \top \sqsubseteq \exists r. \top \geq 1 \rangle$ can lead to cycles in the tableau
- **blocking condition** for $\Pi$-ELC with GCIs: (Bobillo and Straccia 2007)
  $x$ is blocked by $y$ if their assertions and constraints are isomorphic
- the algorithm is not sound
  (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011)

\[ \langle a : A \geq 0.5 \rangle, \langle A \sqsubseteq \exists r.A \geq 1 \rangle, \langle \top \sqsubseteq \neg \exists r. \top \geq 0.1 \rangle \]

\[
0.5 \leq v_{a:A}, \quad v_{a:A} \leq v_{x_1:A} \cdot v_{(a,x_1):r}, \quad v_{(a,x_1):r} \leq 0.9, \\
v_{x_1:A} \leq v_{x_2:A} \cdot v_{(x_1,x_2):r}, \quad v_{(x_1,x_2):r} \leq 0.9, \\
v_{x_2:A} \leq v_{x_3:A} \cdot v_{(x_2,x_3):r}, \quad v_{(x_2,x_3):r} \leq 0.9 \\
0.5 \cdot (\frac{10}{9})^i \leq v_{x_i:A} \leadsto v_{x_7:A} > 1!
\]

- **undecidability results** for variants of $\Pi$-ELC and $\mathbb{L}$-ELC with GCIs
  (Baader and Peñaloza 2011a,b; Cerami and Straccia 2011)
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that
\[ v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k} \? \]
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that

\[v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}\]

Simulate the search tree using a fuzzy ontology:

- encode words \(w \in \Sigma^*\) by numbers
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that
\[ v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k} \,? \]

Simulate the search tree using a fuzzy ontology:

• encode words \(w \in \Sigma^*\) by numbers
• initialize the root node

\[ V = \epsilon, \quad W = \epsilon \]
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that 
\[v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}\]?

Simulate the search tree using a fuzzy ontology:
- encode words \(w \in \Sigma^*\) by numbers
- initialize the root node
- create successors
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that

\[ v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k} \]

Simulate the search tree using a fuzzy ontology:

- encode words \(w \in \Sigma^*\) by numbers
- initialize the root node
- create successors
- concatenate encodings
- transfer to the successors

\[ V = \varepsilon, \; W = \varepsilon \]

\[ V = v_1, \; W = w_1 \]

\[ V = v_p, \; W = w_p \]
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that 
\[v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}\] ?

Simulate the search tree using a fuzzy ontology:

- encode words \(w \in \Sigma^*\) by numbers
- initialize the root node
- create successors
- concatenate encodings
- transfer to the successors

\[
\begin{align*}
V &= \varepsilon, \quad W = \varepsilon \\
V &= v_1, \quad W = w_1 \\
V &= v_p, \quad W = w_p \\
V &= v_1 v_1, \quad W = w_1 w_1 \\
V &= v_1 v_p, \quad W = w_1 w_p \\
&\vdots
\end{align*}
\]
The Post Correspondence Problem

Given pairs of words \((v_1, w_1), \ldots, (v_p, w_p)\) over \(\Sigma = \{1, \ldots, n\}\) with \(n > 1\), is there a non-empty sequence of indices \(i_1 \ldots i_k \in \{1, \ldots, p\}^+\) such that

\[v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}\]

Simulate the search tree using a fuzzy ontology:

- encode words \(w \in \Sigma^*\) by numbers
- initialize the root node
- create successors
- concatenate encodings
- transfer to the successors
- check for solution

\[
\begin{align*}
V = \epsilon, & \quad W = \epsilon \\
r_1 = 1, \quad \cdots, \quad r_p = 1
\end{align*}
\]

\[
\begin{align*}
V = v_1, & \quad W = w_1 \\
r_1 = 1, \quad \cdots, \quad r_p = 1
\end{align*}
\]

\[
\begin{align*}
V = v_1 v_1, & \quad W = w_1 w_1 \\
V = v_1 v_p, & \quad W = w_1 w_p \\
\vdots & \quad \vdots
\end{align*}
\]

\[V = W?\]
Undecidable Fuzzy DLs with GCIs

Consistency is **undecidable** (with crisp GCIs) in ...

<table>
<thead>
<tr>
<th>Undecidable Fuzzy DLs</th>
<th>Constructors</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>NCTL</code> [(\land, T, \exists, \Box)]</td>
<td><code>XEL</code> [(\land, T, \exists, \rightarrow, \bot)]</td>
</tr>
<tr>
<td>Assertions</td>
<td>Crisp</td>
</tr>
</tbody>
</table>
| [starts with \(\Pi\)] | {
| \(\Pi\) \(\top\) |
Consistency is undecidable (with crisp GCIs) in ...

- ... $\otimes$-ELC with $\geq$-assertions and $\otimes$-IEL with $=$-assertions for all continuous t-norms except the Gödel t-norm.
Undecidable Fuzzy DLs with GCIs

Consistency is **undecidable (with crisp GCIs)** in ...

- ... $\otimes$-$\mathcal{ELC}$ with $\geq$-assertions and $\otimes$-$\mathcal{IEL}$ with $=$-assertions for all continuous t-norms except the Gödel t-norm
- ... $\otimes$-$\mathcal{NEL}$ with crisp assertions if $\otimes$ starts with $\check{\sqsubseteq}$

<table>
<thead>
<tr>
<th>Undecidable Fuzzy DLs</th>
<th>$\mathcal{NEL}$</th>
<th>$\mathcal{IEL}$</th>
<th>$\mathcal{ELC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assertions</td>
<td>$\geq$</td>
<td>$=$</td>
<td></td>
</tr>
<tr>
<td>Crisp assertions</td>
<td>starts with $\check{\sqsubseteq}$</td>
<td>starts with $\check{\sqsubseteq}$</td>
<td>$\check{\sqsubseteq}$</td>
</tr>
<tr>
<td></td>
<td>starts with $\check{\sqsubseteq}$</td>
<td>starts with $\check{\sqsubseteq}$</td>
<td>not G</td>
</tr>
<tr>
<td></td>
<td>starts with $\check{\sqsubseteq}$</td>
<td>not G</td>
<td>not G</td>
</tr>
</tbody>
</table>
Consistency is **undecidable** (with crisp GCIs) in ...

- ... $\otimes$-\textit{ELC} with $\geq$-assertions and $\otimes$-\textit{IEL} with $=$-assertions for all continuous $t$-norms except the G"odel $t$-norm.
- ... $\otimes$-\textit{IEL} with crisp assertions if $\otimes$ starts with Ł.
- ... $\Pi$-\textit{ELC} with crisp assertions.

<table>
<thead>
<tr>
<th>undecidable fuzzy DLs</th>
<th>$\textit{IEL}$</th>
<th>$\textit{IEL}$</th>
<th>$\textit{ELC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>assertions</td>
<td>$[\cap, T, \exists, \Box]$</td>
<td>$[\cap, T, \exists, \top, \bot]$</td>
<td>$[\cap, T, \exists, \neg]$</td>
</tr>
<tr>
<td>crisp</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>$\Pi$ Ł</td>
</tr>
<tr>
<td>$\geq$</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>not G</td>
</tr>
<tr>
<td>$=$</td>
<td>starts with Ł</td>
<td>not G</td>
<td>not G</td>
</tr>
</tbody>
</table>
Decidable Fuzzy DLs with GCIs (I)

Consistency is decidable (EXPTIME-complete) in $\otimes\mathcal{E}\mathcal{L}$ with $\geq$-assertions if $\otimes$ does not start with Ł.

- residual negation is crisp: $\neg x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$
- distinguish two "truth values" $\neg x > 0$
- $\exists$-assertions crisp
- starts with Ł
- $\mathcal{C}$ initiates with Ł
- restrict to crisp models of crisp ontologies

$\mathcal{A} = \{C \sqsubseteq D \geq 0, \ldots\}$ is consistent iff $\{C \sqsubseteq D, \ldots\}$ is consistent.

- $\otimes\mathcal{E}\mathcal{L}$ constructors:
  - $\mathcal{N}\mathcal{EL}$ $\mathcal{E}\mathcal{L}$ $\mathcal{C}\mathcal{L}$
  - $[\sqcap, \top, \exists, \neg]$
  - $[\sqcap, \top, \exists, \rightarrow, \bot]$
  - $[\sqcap, \top, \exists, \neg]$
Decidable Fuzzy DLs with GCIs (I)

Consistency is **decidable** (EXPTIME-complete) in $\otimes\mathcal{EL}$ with $\geq$-assertions if $\otimes$ does not start with $\mathcal{L}$

- residual negation is **crisp**: $\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$

- distinguish two “truth values” = 0 and > 0
Decidable Fuzzy DLs with GCIs (I)

Consistency is **decidable (\textsc{Exptime}-complete)** in $\otimes\mathcal{IE}$ with $\geq$-assertions if $\otimes$ does not start with $\bot$

- residual negation is **crisp**: $\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$

- distinguish two "truth values" $= 0$ and $> 0$

\[(C \sqcap D)(x) > 0 \text{ iff } C(x) > 0 \text{ and } D(x) > 0\]

\[(\exists r.C)(x) > 0 \text{ iff there is } y \text{ with } r(x, y) \otimes C(y) > 0\]
Decidable Fuzzy DLs with GCIs (I)

Consistency is decidable (EXPTIME-complete) in $\otimes\mathcal{EL}$ with $\geq$-assertions if $\otimes$ does not start with $\bot$.

- residual negation is crisp: $\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$

- distinguish two “truth values” $= 0$ and $> 0$

$$(C \sqcap D)^\mathcal{I}(x) > 0 \text{ iff } C^\mathcal{I}(x) > 0 \text{ and } D^\mathcal{I}(x) > 0$$

$$(\exists r.C)^\mathcal{I}(x) > 0 \text{ iff there is } y \text{ with } r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y) > 0$$

- restrict to crisp models of crisp ontologies

$$\{\langle C \sqsubseteq D \geq 0.7 \rangle, \ldots \} \text{ is consistent iff } \{C \sqsubseteq D, \ldots \} \text{ is consistent}$$
Decidable Fuzzy DLs with GCIs (I)

Consistency is decidable (EXPTIME-complete) in $\otimes\mathcal{I}\mathcal{E}\mathcal{L}$ with $\geq$-assertions if $\otimes$ does not start with Ł

- residual negation is crisp: $\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$
- distinguish two “truth values” $= 0$ and $> 0$
  
  $(C \sqcap D)^\mathcal{I}(x) > 0$ iff $C^\mathcal{I}(x) > 0$ and $D^\mathcal{I}(x) > 0$
  
  $(\exists r.C)^\mathcal{I}(x) > 0$ iff there is $y$ with $r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y) > 0$

- restrict to crisp models of crisp ontologies

  \[ \{ \langle C \sqcap D \geq 0.7 \rangle, \ldots \} \] is consistent iff \[ \{ C \sqcap D, \ldots \} \] is consistent

- does not work with involutive negation or $=$-assertions
Decidable Fuzzy DLs with GCIs (I)

Consistency is decidable (EXPTIME-complete) in $\otimes$-IEL with $\geq$-assertions if $\otimes$ does not start with Ł

- residual negation is crisp: $\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$

- distinguish two "truth values" = 0 and > 0

\[(C \sqcap D)^I(x) > 0 \text{ iff } C^I(x) > 0 \text{ and } D^I(x) > 0\]
\[(\exists r.C)^I(x) > 0 \text{ iff there is } y \text{ with } r^I(x, y) \otimes C^I(y) > 0\]

- restrict to crisp models of crisp ontologies

\{\langle C \sqsubseteq D \geq 0.7, \ldots \rangle \} is consistent iff \{C \sqsubseteq D, \ldots \} is consistent

- does not work with involutive negation or $=$-assertions

<table>
<thead>
<tr>
<th>undecidable fuzzy DLs</th>
<th>$\Pi EL$</th>
<th>$\I EL$</th>
<th>$ELC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>assertions</td>
<td>constructors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>crisp</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>$\Pi$ Ł</td>
</tr>
<tr>
<td>$\geq$</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>not G</td>
</tr>
<tr>
<td>$=$</td>
<td>starts with Ł</td>
<td>not G</td>
<td>not G</td>
</tr>
</tbody>
</table>
Decidable Fuzzy DLs with GCIs (I)

Consistency is decidable (\(\text{EXPTIME-complete}\)) in \(\otimes-\mathcal{IEL}\) with \(\geq\)-assertions if \(\otimes\) does not start with \(\bot\)

- residual negation is crisp: \(\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}\)
- distinguish two “truth values” = 0 and > 0
  \[(C \sqcap D)^I(x) > 0 \text{ iff } C^I(x) > 0 \text{ and } D^I(x) > 0\]
  \[(\exists r. C)^I(x) > 0 \text{ iff there is } y \text{ with } r^I(x, y) \otimes C^I(y) > 0\]

- restrict to crisp models of crisp ontologies
  \[
  \{(C \sqsubseteq D \geq 0.7), \ldots\} \text{ is consistent iff } \{C \sqsubseteq D, \ldots\} \text{ is consistent}
  \]

- does not work with involutive negation or \(=\)-assertions

<table>
<thead>
<tr>
<th>undecidable fuzzy DLs</th>
<th>(\mathcal{NEL}) [(\sqcap, T, \exists, \sqsubseteq)]</th>
<th>(\mathcal{IEL}) [(\sqcap, T, \exists, \rightarrow, \bot)]</th>
<th>(\mathcal{ELC}) [(\sqcap, T, \exists, \neg)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>crisp assertions</td>
<td>starts with (\ell)</td>
<td>starts with (\ell)</td>
<td>(\ell)</td>
</tr>
<tr>
<td>(\geq)</td>
<td>starts with (\ell)</td>
<td>starts with (\ell)</td>
<td>not (G)</td>
</tr>
<tr>
<td>(=)</td>
<td>starts with (\ell)</td>
<td>not (G)</td>
<td>not (G)</td>
</tr>
</tbody>
</table>
Decidable Fuzzy DLs with GCIs (II)

Consistency is decidable (\textsc{Exptime}-complete) in G-\textsc{Iel} with =-assertions
Decidable Fuzzy DLs with GCIs (II)

Consistency is decidable (ExpTime-complete) in $\mathcal{GELC}$ with $=$-assertions

- Gödel residuum: $x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$

- no finite-valued model property
Decidable Fuzzy DLs with GCIs (II)

Consistency is decidable (EXPTIME-complete) in $G$-IELC with $\equiv$-assertions

- Gödel residuum: $x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
- no finite-valued model property

\[
\langle a: A = 0.5 \rangle \langle T \sqsubseteq \exists r. T \geq 1 \rangle \langle \exists r. A \sqsubseteq B \geq 1 \rangle \langle B \sqsubseteq A \geq 1 \rangle \langle A \rightarrow B \sqsubseteq B \geq 1 \rangle
\]
Decidable Fuzzy DLs with GCIs (II)

Consistency is decidable (EXPTIME-complete) in G-$\mathcal{ELC}$ with $\equiv$-assertions

- Gödel residuum: $x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
- no finite-valued model property

\[
\langle a : A = 0.5 \rangle \langle T \sqsubseteq \exists r.T \geq 1 \rangle \langle \exists r.A \sqsubseteq B \geq 1 \rangle \langle B \sqsubseteq A \geq 1 \rangle \langle A \rightarrow B \sqsubseteq B \geq 1 \rangle
\]

\[
A(x_0) = 0.5 \quad r(x_i, x_{i+1}) = 1 \quad A(x_{i+1}) \leq B(x_i) \quad B(x_i) \leq A(x_i) \quad B(x_i) < A(x_i)
\]
Decidable Fuzzy DLs with GCIs (II)

Consistency is decidable (\textsc{ExpTime}-complete) in G-$\mathcal{IELC}$ with \(=\)-assertions

- Gödel residuum: \(x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}\)
- no finite-valued model property

\[
\langle a : A = 0.5 \rangle \langle T \sqsubseteq \exists r. T \geq 1 \rangle \langle \exists r. A \sqsubseteq B \geq 1 \rangle \langle B \sqsubseteq A \geq 1 \rangle \langle A \rightarrow B \sqsubseteq B \geq 1 \rangle
\]

\[
A(x_0) = 0.5 \quad r(x_i, x_{i+1}) = 1 \quad A(x_{i+1}) \leq B(x_i) \quad B(x_i) \leq A(x_i) \quad B(x_i) < A(x_i)
\]

\[
0.5 = A(x_0) > A(x_1) > A(x_2) > \ldots
\]
Decidable Fuzzy DLs with GCIs (II)

Consistency is \textbf{decidable (ExpTIme-complete)} in G-\(\mathcal{ELC}\) with \(\equiv\)-assertions

- Gödel residuum: \(x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}\)

- no finite-valued model property

\[
\langle a : A = 0.5 \rangle \; \langle T \sqsubseteq \exists r. T \geq 1 \rangle \; \langle \exists r. A \sqsubseteq B \geq 1 \rangle \; \langle B \sqsubseteq A \geq 1 \rangle \; \langle A \rightarrow B \sqsubseteq B \geq 1 \rangle
\]

\begin{align*}
A(x_0) &= 0.5 & r(x_i, x_{i+1}) &= 1 & A(x_{i+1}) &\leq B(x_i) & B(x_i) &\leq A(x_i) & B(x_i) &< A(x_i) \\
0.5 &= A(x_0) > A(x_1) > A(x_2) > \ldots 
\end{align*}

- only the order is relevant
Decidable Fuzzy DLs with GCIs (II)

Consistency is decidable (**EXPTIME-complete**) in G-**IELC** with 

- Gödel residuum: $x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
- no finite-valued model property

$$
\langle a : A = 0.5 \rangle \langle T \sqsubseteq \exists r. T \geq 1 \rangle \langle \exists r. A \sqsubseteq B \geq 1 \rangle \langle B \sqsubseteq A \geq 1 \rangle \langle A \rightarrow B \sqsubseteq B \geq 1 \rangle \\
A(x_0) = 0.5 \quad r(x_i, x_{i+1}) = 1 \quad A(x_{i+1}) \leq B(x_i) \quad B(x_i) \leq A(x_i) \quad B(x_i) < A(x_i) \\
0.5 = A(x_0) > A(x_1) > A(x_2) > \ldots
$$

- only the order is relevant

<table>
<thead>
<tr>
<th>Undecidable Fuzzy DLs</th>
<th>Constructors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>IEC</strong></td>
</tr>
<tr>
<td></td>
<td>$[\land, T, \exists, \sqsubseteq]$</td>
</tr>
<tr>
<td>Assertions</td>
<td>Crisp</td>
</tr>
<tr>
<td></td>
<td>$\geq$</td>
</tr>
<tr>
<td></td>
<td>$=$</td>
</tr>
</tbody>
</table>
Decidable Fuzzy DLs with GCIs (II)

Consistency is **decidable** (**EXPTIME-complete**) in G-\(\mathcal{IELC}\) with \(\equiv\)-assertions

- Gödel residuum: \(x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \)
- no finite-valued model property

\[
\langle a: A = 0.5 \rangle \langle T \sqsubseteq \exists r. T \geq 1 \rangle \langle \exists r. A \sqsubseteq B \geq 1 \rangle \langle B \sqsubseteq A \geq 1 \rangle \langle A \rightarrow B \sqsubseteq B \geq 1 \rangle
\]
\[
A(x_0) = 0.5 \quad r(x_i, x_{i+1}) = 1 \quad A(x_{i+1}) \leq B(x_i) \quad B(x_i) \leq A(x_i) \quad B(x_i) < A(x_i)
\]
\[
0.5 = A(x_0) > A(x_1) > A(x_2) > \ldots
\]

- only the order is relevant

<table>
<thead>
<tr>
<th>undecidable fuzzy DLs</th>
<th>(\mathcal{IEL})</th>
<th>(\mathcal{IEL})</th>
<th>(\mathcal{ELC})</th>
</tr>
</thead>
<tbody>
<tr>
<td>assertions</td>
<td>([\land, T, \exists, \square])</td>
<td>([\land, T, \exists, \rightarrow, \perp])</td>
<td>([\land, T, \exists, \neg])</td>
</tr>
<tr>
<td>crisp</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>Ł</td>
</tr>
<tr>
<td>(\geq)</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>not G</td>
</tr>
<tr>
<td>(\equiv)</td>
<td>starts with Ł</td>
<td>not G</td>
<td>not G</td>
</tr>
</tbody>
</table>

Dresden, 23.05.2014 Fuzzy Description Logics with GCIs
Outline

- Introduction to Fuzzy Description Logics
- Fuzzy Description Logics over $[0, 1]$
- Fuzzy Description Logics over Finite Lattices
- Summary
Complete Residuated De Morgan Lattices

More general:

- complete distributive lattice \((L, \lor, \land, 0, 1)\)
Complete Residuated De Morgan Lattices

More general:
- complete distributive lattice \((L, \lor, \land, 0, 1)\)
- (generalized) t-norm \(\otimes: L \times L \rightarrow L\):
  associative, commutative, monotone, unit 1, ("continuous")

\[
\begin{align*}
\text{1} & \quad \text{0} \\
\text{a} & \quad \text{b} \quad \text{c} \\
\text{happy} & \quad \text{like} \\
\text{laura} & \quad \text{elisabeth} \\
\end{align*}
\]

Dresden, 23.05.2014
Complete Residuated De Morgan Lattices

More general:

- complete distributive lattice \((L, \lor, \land, 0, 1)\)
- (generalized) t-norm \(\otimes: L \times L \rightarrow L:\)
  - associative, commutative, monotone, unit 1, ("continuous")
- residuum \(\Rightarrow: [0, 1] \times [0, 1] \rightarrow [0, 1]: x \otimes y \leq z \text{ iff } y \leq x \Rightarrow z\)
- residual negation \(\ominus x = x \Rightarrow 0\)
Complete Residuated De Morgan Lattices

More general:

- complete distributive lattice \((L, \lor, \land, 0, 1)\)
- (generalized) t-norm \(\otimes : L \times L \rightarrow L\):
  - associative, commutative, monotone, unit 1, ("continuous")
- residuum \(\Rightarrow : [0, 1] \times [0, 1] \rightarrow [0, 1] : x \otimes y \leq z \iff y \leq x \Rightarrow z\)
- residual negation \(\ominus x = x \Rightarrow 0\)
- involutive De Morgan negation \(\sim : L \rightarrow L\)
Complete Residuated De Morgan Lattices

More general:

- complete distributive lattice \((L, \lor, \land, 0, 1)\)
- (generalized) t-norm \(\otimes: L \times L \rightarrow L:\) associative, commutative, monotone, unit 1, ("continuous")
- residuum \(\Rightarrow: [0, 1] \times [0, 1] \rightarrow [0, 1]: x \otimes y \leq z \text{ iff } y \leq x \Rightarrow z\)
- residual negation \(\ominus x = x \Rightarrow 0\)
- involutive De Morgan negation \(\sim: L \rightarrow L\)

\(L\)-\(\mathcal{E}\mathcal{L}\):

- \(\text{Happy}^\mathcal{I}: \Delta^\mathcal{I} \rightarrow L\)
- \(\text{likes}^\mathcal{I}: \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow L\)
- \((\exists r.C)^\mathcal{I}(x) = \bigvee_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)\)
Complete Residuated De Morgan Lattices

More general:
- complete distributive lattice \((L, \lor, \land, 0, 1)\)
- (generalized) t-norm \(\otimes: L \times L \rightarrow L: \)
  associative, commutative, monotone, unit 1, (“continuous”)
- residuum \(\Rightarrow: [0, 1] \times [0, 1] \rightarrow [0, 1]: x \otimes y \leq z \iff y \leq x \Rightarrow z\)
- residual negation \(\ominus x = x \Rightarrow 0\)
- involutive De Morgan negation \(\sim: L \rightarrow L\)

\(L\-\mathcal{EL}\) :
- \(\text{Happy}^\mathcal{I}: \Delta^\mathcal{I} \rightarrow L\)
- \(\text{likes}^\mathcal{I}: \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow L\)
- \((\exists r.C)^\mathcal{I}(x) = \bigvee_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)\)

\(\langle\text{laura:Happy = }\sim a\rangle, \langle\text{elisabeth:Happy = }\sim b\rangle\)
Reasoning over Finite Lattices

Reduction to classical reasoning for $L$-ELC with GCIs over finite total orders $L$:
(Bobillo, Delgado, et al. 2012; Straccia 2006)

- introduce cut-concepts and -roles $A_p, r_p$ for every $p \in L$
- $A_p \equiv \text{all individuals } x \text{ with } A^\pi(x) \geq p$

• recursive translation of concepts using cut-concepts
• exponential in the size of $O \Rightarrow 2\text{-E}XP \Rightarrow \text{TIME}$

Satisfiability in $Ł_n$-ELC is $\text{PSPACE}$-complete (without GCIs)
(Bou, Cerami, and Esteva 2011)
Reasoning over Finite Lattices

Reduction to classical reasoning for $L$-$\mathcal{ELC}$ with GCIs over finite total orders $L$:

(Bobillo, Delgado, et al. 2012; Straccia 2006)

- Introduce cut-concepts and -roles $A_p, r_p$ for every $p \in L$
- $A_p \equiv$ all individuals $x$ with $A^\sqcap(x) \geq p$
- $A_{0.5} \sqsubseteq A_{0.25}, r_{0.5} \sqsubseteq r_{0.25}, \ldots$ — needs role hierarchy ($\mathcal{H}$)!
Reasoning over Finite Lattices

Reduction to classical reasoning for $L$-$\mathcal{ELC}$ with GCI$s$ over finite total orders $L$:
(Bobillo, Delgado, et al. 2012; Straccia 2006)

- introduce cut-concepts and -roles $A_p, r_p$ for every $p \in L$
- $A_p \equiv$ all individuals $x$ with $A^x(x) \geq p$
- $A_{0.5} \sqsubseteq A_{0.25}, r_{0.5} \sqsubseteq r_{0.25}, \ldots$ — needs role hierarchy ($\mathcal{H}$)!
- recursive translation of concepts using cut-concepts
Reduction to classical reasoning for $L$-$\mathcal{ELC}$ with GCIIs over finite total orders $L$:

- Introduce cut-concepts and -roles $A_p$, $r_p$ for every $p \in L$
- $A_p \equiv \{ \text{all individuals } x \text{ with } A^\mathfrak{I}(x) \geq p \}$
- $A_{0.5} \sqsubseteq A_{0.25}$, $r_{0.5} \sqsubseteq r_{0.25}$, $\ldots$ — needs role hierarchy ($\mathcal{H}$)!
- Recursive translation of concepts using cut-concepts
- Exponential in the size of $\mathcal{O}$ $\sim 2\text{-EXPTIME}$
Reasoning over Finite Lattices

Reduction to classical reasoning for $L$-$\mathcal{ELC}$ with GCI$s$ over finite total orders $L$:

- introduce cut-concepts and -roles $A_p$, $r_p$ for every $p \in L$
- $A_p \equiv$ all individuals $x$ with $A^\mathcal{I}(x) \geq p$
- $A_{0.5} \sqsubseteq A_{0.25}$, $r_{0.5} \sqsubseteq r_{0.25}$, $\ldots$ — needs role hierarchy $(\mathcal{H})$!
- recursive translation of concepts using cut-concepts
- exponential in the size of $\mathcal{O} \rightsquigarrow 2\text{-EXP TIME}$

Satisfiability in $L_n$-$\mathcal{ELC}$ is $\text{PSPACE}$-complete (without GCI$s$)

(Bou, Cerami, and Esteva 2011)
Automata-Based Approach for Satisfiability

Satisfiability in $L$-\textit{IELC} over any finite $L$ is ...

- ... \textit{EXPTIME}-complete with GCIs
- ... \textit{PSPACE}-complete without GCIs

Adaptation of classical construction (Baader, Hladik, and Peñaloza 2008)

• recognize tree-shaped models by looping tree automaton of exponential size
• states: mappings giving degrees to all relevant concepts
• transition relation: satisfy the semantics of $\exists \, r.C$

• \textit{PSpace}-upper bound under certain conditions
• extension to consistency via pre-completion (Hollunder 1996)

$\exists \, \text{likes} \cdot \exists \, \text{has-disease} \cdot \top \mapsto \rightarrow a \cdot \exists \, \text{has-disease} \cdot \top \mapsto \rightarrow 0, \ldots$

$\exists \, \text{has-disease} \cdot \top \mapsto \rightarrow \sim a \cdot \varphi \mapsto \rightarrow \sim b, \ldots$

...
Automata-Based Approach for Satisfiability

Satisfiability in $L$-$\mathcal{ELC}$ over any finite $L$ is ...

- EXPTIME-complete with GCIs
- PSPACE-complete without GCIs

Adaptation of classical construction (Baader, Hladik, and Peñaloza 2008)

- recognize tree-shaped models by looping tree automaton of exponential size
Automata-Based Approach for Satisfiability

Satisfiability in $L$-$\mathcal{I}ELC$ over any finite $L$ is ...

- $\ldots$ \textit{EXPTIME}-complete with GCIs
- $\ldots$ \textit{PSPACE}-complete without GCIs

Adaptation of classical construction \hfill (Baader, Hladik, and Peñaloza 2008)

- recognize \textit{tree-shaped models} by looping tree automaton of exponential size
- states: mappings giving degrees to all relevant concepts

$\exists \text{likes}. \exists \text{has-disease}. \top \mapsto a, \exists \text{has-disease}. \top \mapsto 0, \ldots$
Automata-Based Approach for Satisfiability

Satisfiability in $L$-$\mathcal{EELC}$ over any finite $L$ is ...

- ... $\text{EXPTIME}$-complete with GCIs
- ... $\text{PSPACE}$-complete without GCIs

Adaptation of classical construction \hspace{1cm} (Baader, Hladik, and Peñaloza 2008)

- recognize tree-shaped models by looping tree automaton of exponential size
- states: mappings giving degrees to all relevant concepts
- transition relation: satisfy the semantics of $\exists r.C$

\[
\begin{align*}
\exists \text{likes}. \exists \text{has-disease}. T & \mapsto a, \exists \text{has-disease}. T \mapsto 0, \ldots \\
\exists \text{has-disease}. T & \mapsto \sim a, \varrho \mapsto \sim b, \ldots, \varrho \mapsto 0, \ldots
\end{align*}
\]
Automata-Based Approach for Satisfiability

Satisfiability in $L$-$\mathcal{EELC}$ over any finite $L$ is ...

- ... $\text{ExpTime}$-complete with GCIs
- ... $\text{PSpace}$-complete without GCIs

Adaptation of classical construction (Baader, Hladik, and Peñaloza 2008)

- recognize tree-shaped models by looping tree automaton of exponential size
- states: mappings giving degrees to all relevant concepts
- transition relation: satisfy the semantics of $\exists r.C$
- $\text{PSpace}$ upper bound under certain conditions
Automata-Based Approach for Satisfiability

Satisfiability in \( L\-\mathcal{ELC} \) over any finite \( L \) is...

- ... \( \text{EXPTIME} \)-complete with GCIs
- ... \( \text{PSPACE} \)-complete without GCIs

Adaptation of classical construction \((\text{Baader, Hladik, and Peñaloza 2008})\)

- recognize tree-shaped models by looping tree automaton of exponential size
- states: mappings giving degrees to all relevant concepts
- transition relation: satisfy the semantics of \( \exists r. C \)
- \( \text{PSPACE} \) upper bound under certain conditions
- extension to consistency via \text{pre-completion} \((\text{Hollunder 1996})\)

\[
\exists \text{likes}. \exists \text{has-disease}. \top \mapsto a, \exists \text{has-disease}. \top \mapsto 0, \ldots
\]

\[
\exists \text{has-disease}. \top \mapsto \sim a, \varrho \mapsto \sim b, \ldots
\]

\[
\varrho \mapsto 0, \ldots
\]
Outline

- Introduction to Fuzzy Description Logics
- Fuzzy Description Logics over $[0, 1]$
- Fuzzy Description Logics over Finite Lattices
- Summary
Summary

Fuzzy DLs with GCIs over [0, 1] often undecidable or inexpressive:

Summary

Fuzzy DLs with GCIs over $[0, 1]$ often **undecidable or inexpressive**:


**Tight complexity results** for fuzzy DLs over finite lattices:

Open Questions

For some cases decidability still unknown

<table>
<thead>
<tr>
<th>undecidable fuzzy DLs</th>
<th>( \mathcal{NEL} ) ([\land, T, \exists, \Box])</th>
<th>( \mathcal{IEL} ) ([\land, T, \exists, \rightarrow, \bot])</th>
<th>( \mathcal{ELC} ) ([\land, T, \exists, \neg])</th>
</tr>
</thead>
<tbody>
<tr>
<td>assertions</td>
<td>crisp</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
</tr>
<tr>
<td>≥</td>
<td>starts with Ł</td>
<td>starts with Ł</td>
<td>not G</td>
</tr>
<tr>
<td>=</td>
<td>starts with Ł</td>
<td>not G</td>
<td>not G</td>
</tr>
</tbody>
</table>

Extension of the decidability results to more expressive DLs

Tight complexity results without GCIs over \([0, 1]\)

In \( \mathcal{EL} \), consistency is trivial, but what about subsumption?

Open Questions

For some cases decidability still unknown

<table>
<thead>
<tr>
<th>Undecidable Fuzzy DLs</th>
<th>( \mathcal{NEL} ) [( \sqcap, T, \exists, \Box )]</th>
<th>( \mathcal{IEL} ) [( \sqcap, T, \exists, \rightarrow, \bot )]</th>
<th>( \mathcal{ELC} ) [( \sqcap, T, \exists, \neg )]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisp assertions</td>
<td>starts with ( \mathcal{L} )</td>
<td>starts with ( \mathcal{L} )</td>
<td>( \top )</td>
</tr>
<tr>
<td>( \geq ) assertions</td>
<td>starts with ( \mathcal{L} )</td>
<td>starts with ( \mathcal{L} )</td>
<td>not ( G )</td>
</tr>
<tr>
<td>( = ) assertions</td>
<td>starts with ( \mathcal{L} )</td>
<td>not ( G )</td>
<td>not ( G )</td>
</tr>
</tbody>
</table>

Extension of the decidability results to more expressive DLs
### Open Questions

For some cases decidability still unknown

<table>
<thead>
<tr>
<th>undecidable fuzzy DLs</th>
<th>constructors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \mathcal{NEL} )</td>
</tr>
<tr>
<td>[( \cap, \top, \exists, \Box )]</td>
<td>[( \cap, \top, \exists, \rightarrow, \bot )]</td>
</tr>
<tr>
<td>assertions</td>
<td>starts with ( \mathcal{L} )</td>
</tr>
<tr>
<td>crisp</td>
<td>( \prod ) ( \mathcal{L} )</td>
</tr>
<tr>
<td>( \geq )</td>
<td>starts with ( \mathcal{L} )</td>
</tr>
<tr>
<td>( = )</td>
<td>starts with ( \mathcal{L} )</td>
</tr>
</tbody>
</table>

Extension of the decidability results to more expressive DLs

Tight complexity results without GCIs over \([0, 1]\)
Open Questions

For some cases decidability still unknown

<table>
<thead>
<tr>
<th>Undecidable Fuzzy DLs</th>
<th>Constructors</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>( \mathcal{NEL} )</strong> ([\top, \bot, \exists, \sqcap] )</td>
<td><strong>( \mathcal{IEL} )</strong> ([\top, \bot, \exists, \rightarrow] )</td>
</tr>
<tr>
<td>Crisp starts with ( \bot )</td>
<td>Starts with ( \bot )</td>
</tr>
<tr>
<td>( \geq ) starts with ( \bot )</td>
<td>Starts with ( \bot )</td>
</tr>
<tr>
<td>( = ) starts with ( \bot )</td>
<td>Not ( G )</td>
</tr>
</tbody>
</table>

Extension of the decidability results to more expressive DLs

Tight complexity results without GCIs over \([0, 1]\)

In \( \mathcal{EL} \), consistency is trivial, but what about subsumption?

Thank you!

\[ \langle x : \exists \text{has.} \text{Question} \geq 0.7 \rangle? \]
References I


---


---


