TECHNISCHE UNIVERSITÄT DRESDEN
DRESDEN concept
Institut für Theoretische Informatik Lehrstuhl für Automatentheorie

FUZZY DESCRIPTION LOGICS WITH GENERAL CONCEPT INCLUSIONS

Verteidigung der Dissertation

Stefan Borgwardt

Dresden, 23.05.2014

Outline

- Introduction to Fuzzy Description Logics
- Fuzzy Description Logics over [0, 1]
- Fuzzy Description Logics over Finite Lattices
- Summary

Motivation

$\mathrm{laura}{:}\mathsf{Human}$

$\mathrm{laura}{:}\mathsf{Female}$

$\mathrm{laura}{:}\mathsf{Happy}$

$(\mathrm{laura}, 2){:}\mathsf{has\text{-}age}$

$\mathrm{laura}{:}\exists\mathsf{sits\text{-}on}.\mathsf{Swing}$

$\mathrm{elisabeth}{:}\mathsf{Human}$

$\mathrm{elisabeth}{:}\mathsf{Female}$

$\mathrm{elisabeth}{:}\mathsf{Happy}$

$(\mathrm{elisabeth}, 4){:}\mathsf{has\text{-}age}$

$\mathrm{elisabeth}{:}\exists\mathsf{wears}.\mathsf{Hood}$

$(\mathrm{laura}, \mathrm{elisabeth}){:}\mathsf{has\text{-}sister}$

$(\mathrm{laura}, \mathrm{elisabeth}){:}\mathsf{likes}$

$\mathsf{Human} \sqcap \exists\mathsf{has\text{-}age}.(\leq 12) \sqsubseteq \mathsf{Child}$

$\mathsf{Child} \sqcap \mathsf{Female} \sqsubseteq \mathsf{Girl}$

Motivation

laura : Human

laura : Female

laura : Happy

(laura, 2) : has-age

laura : ∃sits-on.Swing

elisabeth : Human

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(elisabeth, 4) : has-age

elisabeth : ∃wears.Hood

(laura, elisabeth) : has-sister

(laura, elisabeth) : likes

$\text{Human} \sqcap \exists\text{has-age}.(\leq 12) \sqsubseteq \text{Child}$

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... holds to degree / probability / possibility 0 ... 0.1 ... 0.5 ... 0.9 ... 1

Uncertainty and Vagueness

$\text{laura}:\exists\textsf{sits-on}.\textsf{Swing}$

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Probabilistic/Possibilistic DLs:

- probability/possibility distributions on possible worlds

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- Prob-$\mathcal{ALC}$ (Lutz and Schröder 2010)
- $\mathcal{ALCN}$ with possibilistic axioms (Hollunder 1995)

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Fuzzy DLs: (Straccia 1998; Tresp and Molitor 1998; Yen 1991)

- two degrees of truth (*false*, *true*) are replaced by $[0, 1]$ (Zadeh 1965)

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- two degrees of truth (*false*, *true*) are replaced by $[0,1]$ (Zadeh 1965)
- statements are assigned a truth degree
- conjunction, etc. are interpreted by appropriate truth functions

Mathematical Fuzzy Logic

$$\mathrm{laura}:\mathsf{Happy} \sqcap \mathsf{Cute}$$

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- involutive negation $\sim x := 1 - x$
- t-conorm $x \oplus y := \sim(\sim x \otimes \sim y)$

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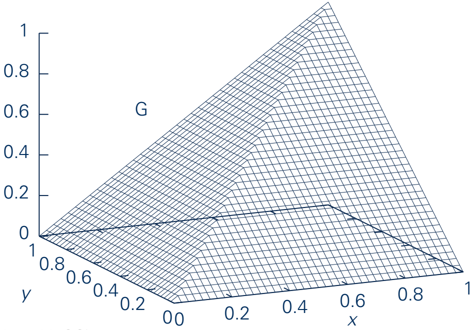

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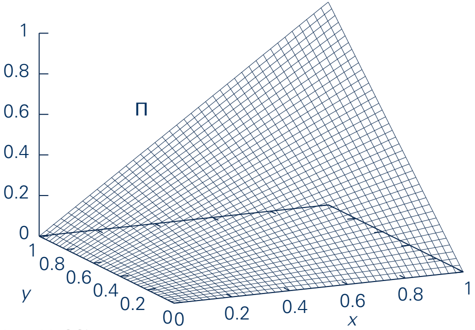

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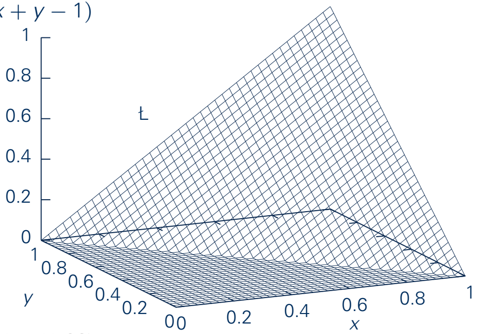

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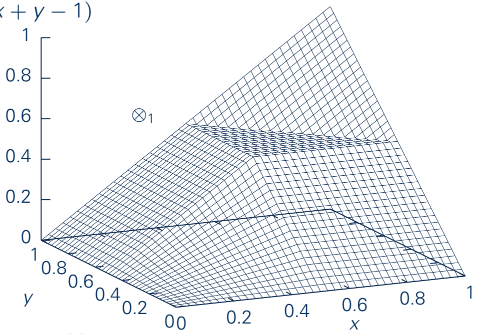

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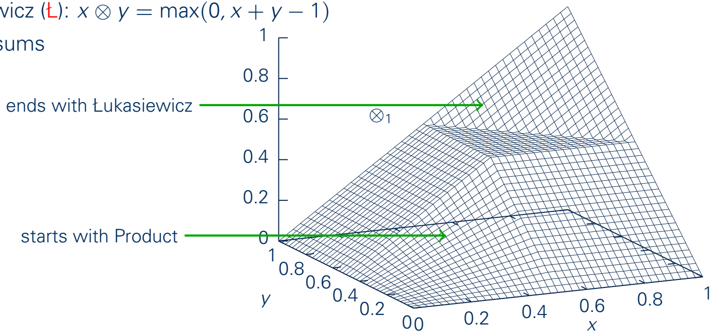

Fuzzy Description Logics

Fuzzy interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

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$\mathcal{EL}$:

- conjunction $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$
- top $\top^\mathcal{I}(x) = 1$
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More constructors:

- $\mathcal{C}$: involutive negation $\neg C$
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$\underbrace{\otimes\text{-}\mathfrak{I}\mathcal{ELC} \quad \Pi\text{-}\mathfrak{N}\mathcal{EL} \quad Ł\text{-}\mathcal{ELC}}_{\text{fuzzy extensions of } \mathcal{ALC}}$ $\quad \otimes\text{-}\mathfrak{I}\mathcal{SCHOI}$

Fuzzy Reasoning

Ontology $\mathcal{O}$: finite set of axioms:

- concept assertion $\langle a : C \triangleright p \rangle$: $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright p$
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Reasoning tasks:

- ontology consistency:
  Does $\mathcal{O}$ have a (witnessed) model?
- concept satisfiability:
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Applications: (Ciaramella et al. 2010; Meghini, Sebastiani, and Straccia 2001)

- recommender systems with background knowledge
- information retrieval, query relaxation

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Tableau algorithm for $\otimes$-$\mathfrak{I}\mathcal{ELC}$ without GCIs: (Bobillo and Straccia 2009)

$$\langle a{:}C \geq p\rangle \rightsquigarrow \langle a{:}C = v_{a:C}\rangle,\ v_{a:C} \geq p$$

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- deterministic exponential time
- $\mathcal{O}$ is consistent iff the constraints have a solution (NP-hard)
- NExpTime for Ł-$\mathcal{ELC}$, ExpSpace for **Π**-$\mathcal{ELC}$
- possible for any finite ordinal sum

Fuzzy GCIs

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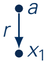

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$$0.5 \leq v_{a:A}, \quad v_{a:A} \leq v_{x_1:A} \cdot v_{(a,x_1):r}, \quad v_{(a,x_1):r} \leq 0.9,$$

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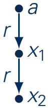

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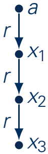

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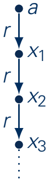

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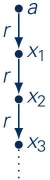

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- undecidability results for variants of Π-$\mathcal{ELC}$ and Ł-$\mathcal{ELC}$ with GCIs
  (Baader and Peñaloza 2011a,b; Cerami and Straccia 2011)

The Post Correspondence Problem

Given pairs of words $(v_1, w_1), \ldots, (v_p, w_p)$ over $\Sigma = \{1, \ldots, n\}$ with $n > 1$, is there a non-empty sequence of indices $i_1 \ldots i_k \in \{1, \ldots, p\}^+$ such that $v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}$?

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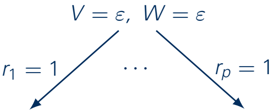

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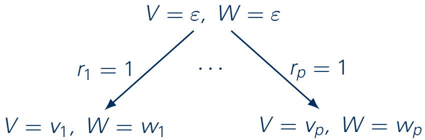

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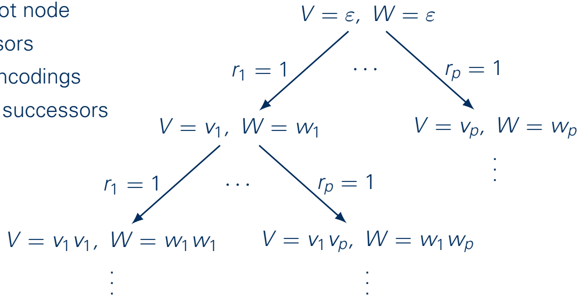

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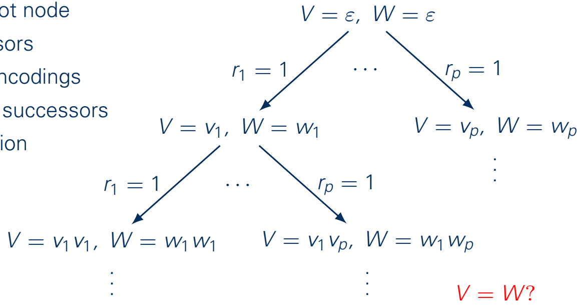
- check for solution



Undecidable Fuzzy DLs with GCIs

Consistency is undecidable (with crisp GCIs) in ...

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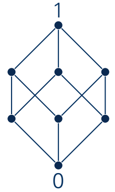
- Fuzzy Description Logics over [0, 1]
- Fuzzy Description Logics over Finite Lattices
- Summary

Complete Residuated De Morgan Lattices

More general:

- complete distributive lattice $(L, \vee, \wedge, 0, 1)$



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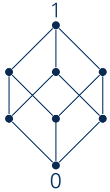

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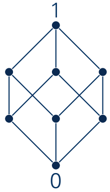

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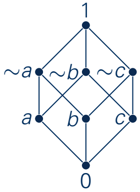

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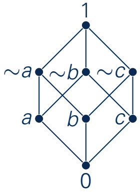

L-$\mathcal{EL}$:

- $\text{Happy}^\mathcal{I}\colon \Delta^\mathcal{I} \to L$
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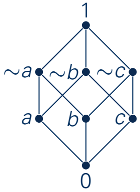

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$\langle \mathrm{laura}\!:\!\text{Happy} = {\sim}a \rangle$, $\langle \mathrm{elisabeth}\!:\!\text{Happy} = {\sim}b \rangle$

Reasoning over Finite Lattices

Reduction to classical reasoning for L-$\mathcal{ELC}$ with GCIs over finite total orders L:
(Bobillo, Delgado, et al. 2012; Straccia 2006)

- introduce cut-concepts and -roles A_p, r_p for every $p \in L$
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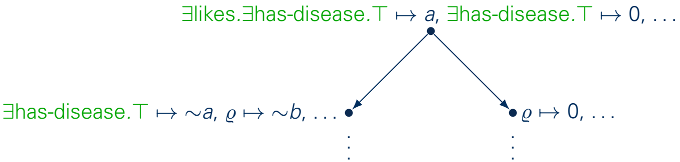

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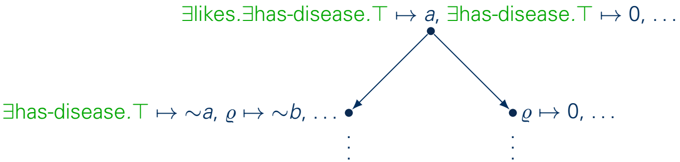

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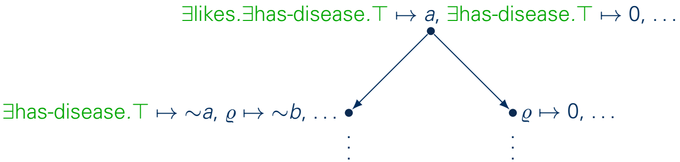
- extension to consistency via pre-completion (Hollunder 1996)



Outline

- Introduction to Fuzzy Description Logics
- Fuzzy Description Logics over [0, 1]
- Fuzzy Description Logics over Finite Lattices
- Summary

Summary

Fuzzy DLs with GCIs over [0, 1] often undecidable or inexpressive:

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- B, Felix Distel, Rafael Peñaloza (2014). "The Limits of Decidability in Fuzzy Description Logics with General Concept Inclusions". In: *Artif. Intell.* Under submission.
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Tight complexity results for fuzzy DLs over finite lattices:

- B, Rafael Peñaloza (2011). "Description Logics over Lattices with Multi-valued Ontologies". In: *Proc. IJCAI'11*, pages 768–773.
- B, Rafael Peñaloza (2012). "A Tableau Algorithm for Fuzzy Description Logics over Residuated De Morgan Lattices". In: *Proc. RR'12*. Volume 7497. LNCS, pages 9–24.
- B, Rafael Peñaloza (2013). "The Complexity of Lattice-Based Fuzzy Description Logics". In: *J. Data Semant.* 2.1, pages 1–19.
- B, Rafael Peñaloza (2014). "Consistency Reasoning in Lattice-Based Fuzzy Description Logics". In: *Int. J. Approx. Reason.* In Press.

Open Questions

For some cases decidability still unknown

undecidable fuzzy DLs		constructors $\mathcal{NEL}$ $[\sqcap, \top, \exists, \boxminus]$	constructors $\mathcal{IEL}$ $[\sqcap, \top, \exists, \to, \bot]$	constructors $\mathcal{ELC}$ $[\sqcap, \top, \exists, \neg]$
assertions	crisp	starts with Ł	starts with Ł	Π Ł
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Extension of the decidability results to more expressive DLs

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In $\mathcal{EL}$, consistency is trivial, but what about subsumption?

- B, Rafael Peñaloza. "Positive Subsumption in Fuzzy $\mathcal{EL}$ with General t-norms". In: *Proc. IJCAI'13*, pages 789–795.

Thank you!

$$\langle x : \exists \text{has.Question} \geq 0.7 \rangle ?$$

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