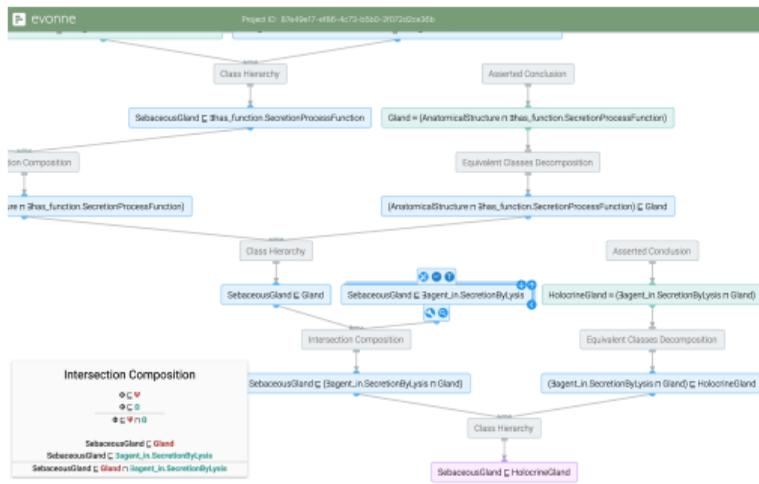


Christian Alrabbaa, Stefan Borgwardt, Patrick Koopmann, Alisa Kovtunova
Technische Universität Dresden, Center for Perspicuous Computing (CPEC)

Finding Good Proofs for Ontology-Mediated Query Answers

(extended abstract of DL paper) // XLoKR'22, Haifa, Israel, 31st July 2022

Motivation



Ontologies, Proofs

General Settings

- Shortening
 - Method
 - Can't case shortening
- Proof Settings
 - Shortening
 - Shorten Public Names
 - Shorten All**
 - Compactness
 - Display Allowed
 - Behavior
 - Full Position
 - Left Bottom Corner
 - Linear Proof
 - Structure Mode
 - Magic Hide
- Ontology Settings

Motivation

The screenshot shows the Evolve tool interface. At the top, there is a header bar with the project name 'evolve' and a project ID. Below the header, there are two main sections: 'Class Hierarchy' and 'Asserted Conclusion'. The 'Class Hierarchy' section shows a tree structure with nodes like 'SubclassOf(Gland, C, this, function, Secretion, Process, Function)' and 'Gland = (AnatomicalStructure in this, function, Secretion, Process, Function)'. The 'Asserted Conclusion' section shows a table of employee data. The table has columns: empno, ename, job, mgr, hiredate, sal, comm, deptno. The 'job' column is highlighted in blue for the first row. On the left side, there is a sidebar with a tree structure showing 'ion-Composition', 'Inte', and 'SubclassOf(Gland, Secretion, Process, Function)'. On the right side, there is a 'General Settings' panel with options for 'Shortening' and 'Cancel case shortening'.

empno	ename	job	mgr	hiredate	sal	comm	deptno
7369	SMITH	CLERK	7902	1980-12-17	800.00	(Null)	20
7499	ALLEN	SALESMAN	7698	1981-02-20	1600.00	300.00	30
7521	WARD	SALESMAN	7698	1981-02-22	1250.00	500.00	30
7566	JONES	MANAGER	7839	1981-04-02	2975.00	(Null)	20
7654	MARTIN	SALESMAN	7698	1981-09-28	1250.00	1400.00	30
7698	BLAKE	MANAGER	7839	1981-05-01	2850.00	(Null)	30
7782	CLARK	MANAGER	7839	1981-06-09	2450.00	(Null)	10
7788	SCOTT	ANALYST	7566	1987-07-13	3000.00	(Null)	20
7839	KING	PRESIDENT	(Null)	1981-11-07	5000.00	(Null)	10
7844	TURNER	SALESMAN	7698	1981-09-08	1500.00	0.00	30
7876	ADAMS	CLERK	7788	1987-07-13	1100.00	(Null)	20
7900	JAMES	CLERK	7698	1981-12-03	950.00	(Null)	30
7902	FORD	ANALYST	7566	1981-12-03	3000.00	(Null)	20

Ontologies, Proofs

plus Data, Queries

Motivation

The screenshot shows the evonone tool interface. At the top, there is a header with the project ID: 67e49e17-e486-4c73-b5d0-2107220a336a. Below the header, there are two main sections: "Class Hierarchy" and "Asserted Conclusion". The "Class Hierarchy" section shows a tree structure with nodes like "SubaceousGland", "Gland", and "Inte". The "Asserted Conclusion" section shows a table of employee data. The table has columns: empno, ename, job, mgr, hiredate, sal, comm, deptno. The data is as follows:

empno	ename	job	mgr	hiredate	sal	comm	deptno
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7782	CLARK				2450.00	(Null)	10
7788	SCOTT				3000.00	(Null)	20
7839	KING				5000.00	(Null)	10
7844	TURKI				1500.00	0.00	30
7876	ADA				1100.00	(Null)	20
7900	JAM				950.00	(Null)	30
7902	FORI				3000.00	(Null)	20

Overlaid on the table is a diagram labeled T and α . The diagram shows a network of nodes and arrows. A large rounded rectangle labeled T encloses several nodes. An arrow points from a node inside T to a node labeled α outside the rectangle. There are also arrows between nodes inside T .

Ontologies, Proofs

plus Data, Queries

Algorithms, Complexity

Outline

Preliminaries

- Description Logics

- Proofs

- Ontology-Mediated Queries

Proof Systems

- Deriving CQs

- Deriving Ground Atoms

Complexity

Summary

Description Logics (DLs)

DLs are fragments of first-order logic with a funny syntax.

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A DL \mathcal{T} Box/ontology \mathcal{T} :

existential rules (implicit \forall omitted)

DL syntax

$\text{Spoke}(x) \rightarrow \exists y. \text{partOf}(x, y) \wedge \text{Wheel}(y)$

$\text{Spoke} \sqsubseteq \exists \text{partOf}. \text{Wheel}$

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$\text{partOf}(x, y) \rightarrow \text{hasPart}(y, x)$	$\text{partOf} \sqsubseteq \text{hasPart}^{-}$
$\text{hasPart}(x, y) \rightarrow \exists z. \text{attachedTo}(y, z)$	$\exists \text{hasPart}^{-} \sqsubseteq \exists \text{attachedTo}$
$\text{Wheel}(x) \rightarrow \exists y. \text{partOf}(x, y) \wedge \text{Bike}(y)$	$\text{Wheel} \sqsubseteq \exists \text{partOf}. \text{Bike}$

Description Logics (DLs) DL-Lite_R

DLs are fragments of first-order logic with a funny syntax.

A DL DL-Lite_R TBox/ontology \mathcal{T} :

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DL syntax

$\text{Spoke}(x) \rightarrow \exists y. \text{partOfW}(x, y)$

$\text{Spoke} \sqsubseteq \exists \text{partOfW}$

$\text{partOf}(x, y) \rightarrow \text{hasPart}(y, x)$

$\text{partOf} \sqsubseteq \text{hasPart}^-$

$\text{hasPart}(x, y) \rightarrow \exists z. \text{attachedTo}(y, z)$

$\exists \text{hasPart}^- \sqsubseteq \exists \text{attachedTo}$

$\text{Wheel}(x) \rightarrow \exists y. \text{partOfB}(x, y)$

$\text{Wheel} \sqsubseteq \exists \text{partOfB}$

$\text{partOfW}(x, y) \rightarrow \text{partOf}(x, y)$

$\text{partOfW} \sqsubseteq \text{partOf}$

$\text{partOfW}(x, y) \rightarrow \text{Wheel}(y)$

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$\text{partOfB}(x, y) \rightarrow \text{partOf}(x, y)$

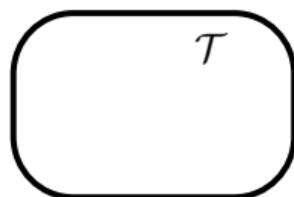
$\text{partOfB} \sqsubseteq \text{partOf}$

$\text{partOfB}(x, y) \rightarrow \text{Bike}(y)$

$\exists \text{partOfB}^- \sqsubseteq \text{Bike}$

Entailment: \mathcal{T} implies a new sentence α , e.g.,

$$\text{partOfW}(x, y) \rightarrow \exists z. \text{partOfB}(y, z) \quad \exists \text{partOfW}^- \sqsubseteq \exists \text{partOfB}$$



Π
 α

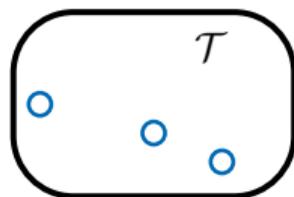
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Justification: Minimal subset of \mathcal{T} entailing α



Π
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Proofs

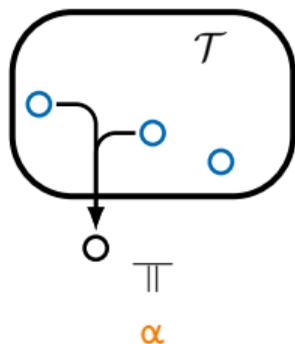
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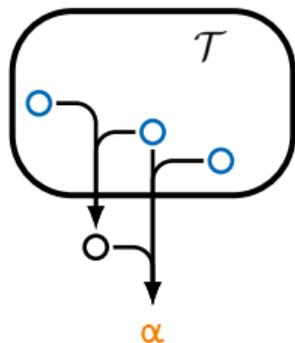
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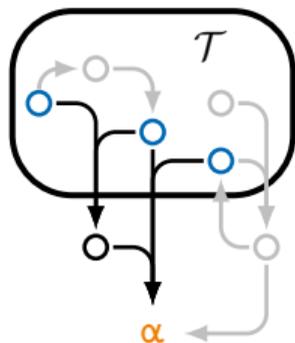
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Deriver: Provides hypergraphs $\mathcal{D}(\mathcal{T}, \alpha)$ of all permissible inference steps for $\mathcal{T} \models \alpha$



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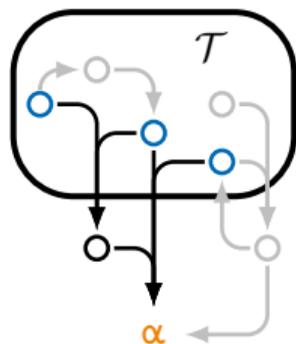
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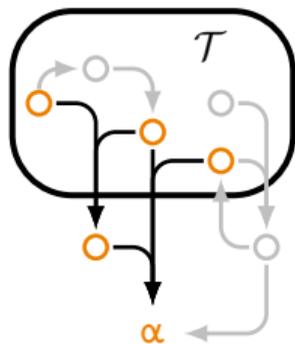
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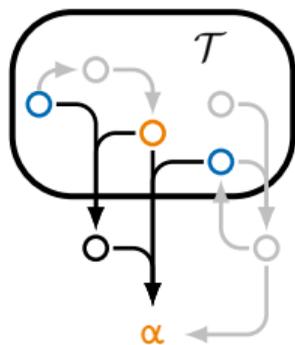
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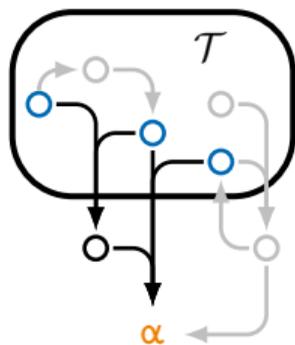
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Decision problem: Decide whether there exists a proof of measure $\leq n$



Ontology-Mediated Queries

Ontology-mediated queries generalize database queries

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ABox \mathcal{A} : Spoke(a), partOf(a, b), ...

Ontology-Mediated Queries

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Conjunctive query $q(x)$: $\exists y, z, v, w, w'$.

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What is the complexity of **explaining** this entailment?

Outline

Preliminaries

- Description Logics

- Proofs

- Ontology-Mediated Queries

Proof Systems

- Deriving CQs

- Deriving Ground Atoms

Complexity

Summary

Deriving CQs (\mathcal{D}_{cq})

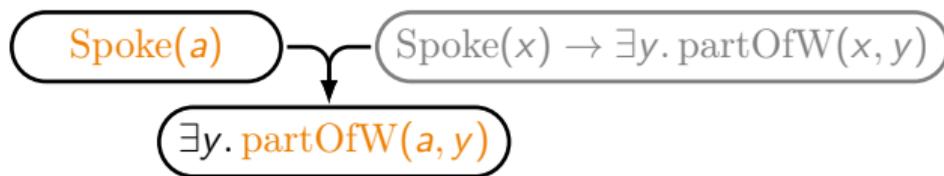
(Stefanoni 2011; Croce and Lenzerini 2018)

What are the **inference rules** for proving $\mathcal{T} \cup \mathcal{A} \models q(a)$ (in $DL-Lite_R$)?

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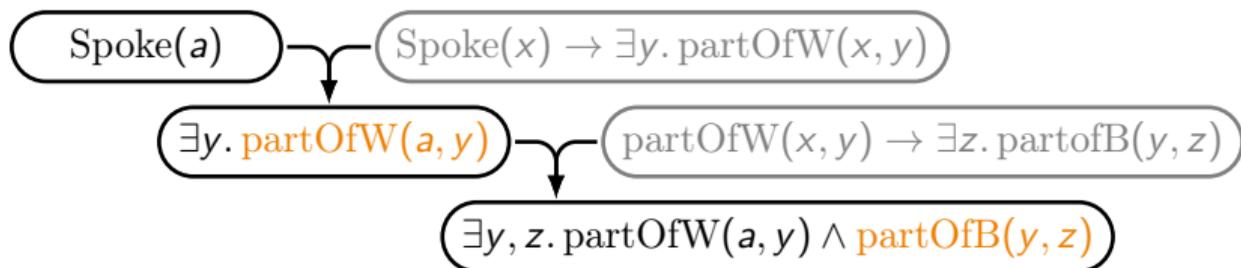
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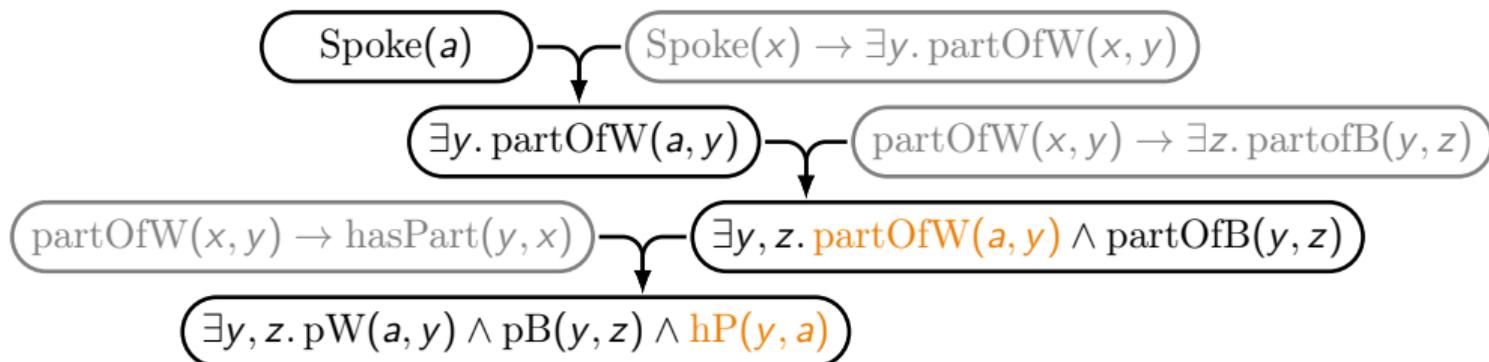
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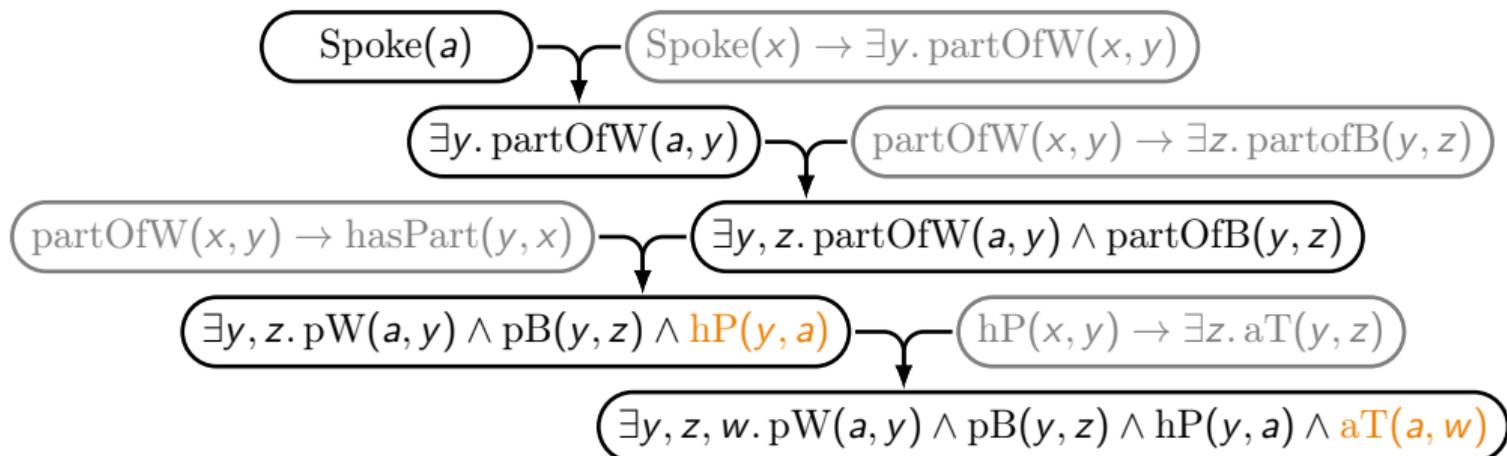
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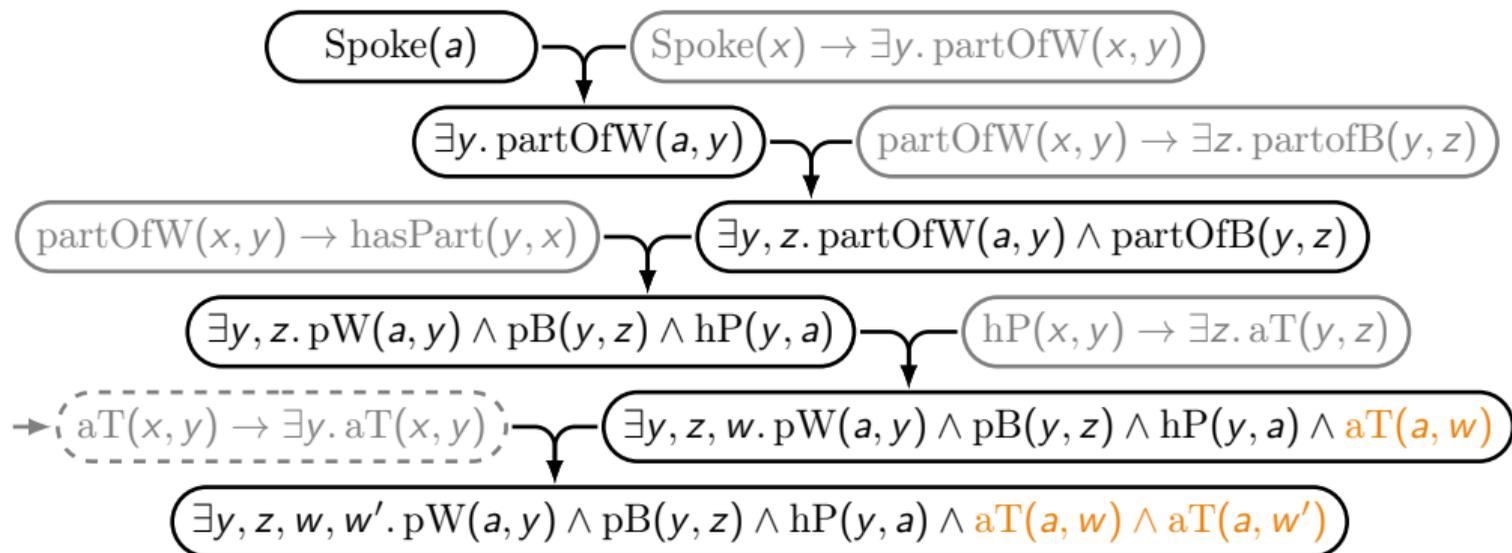
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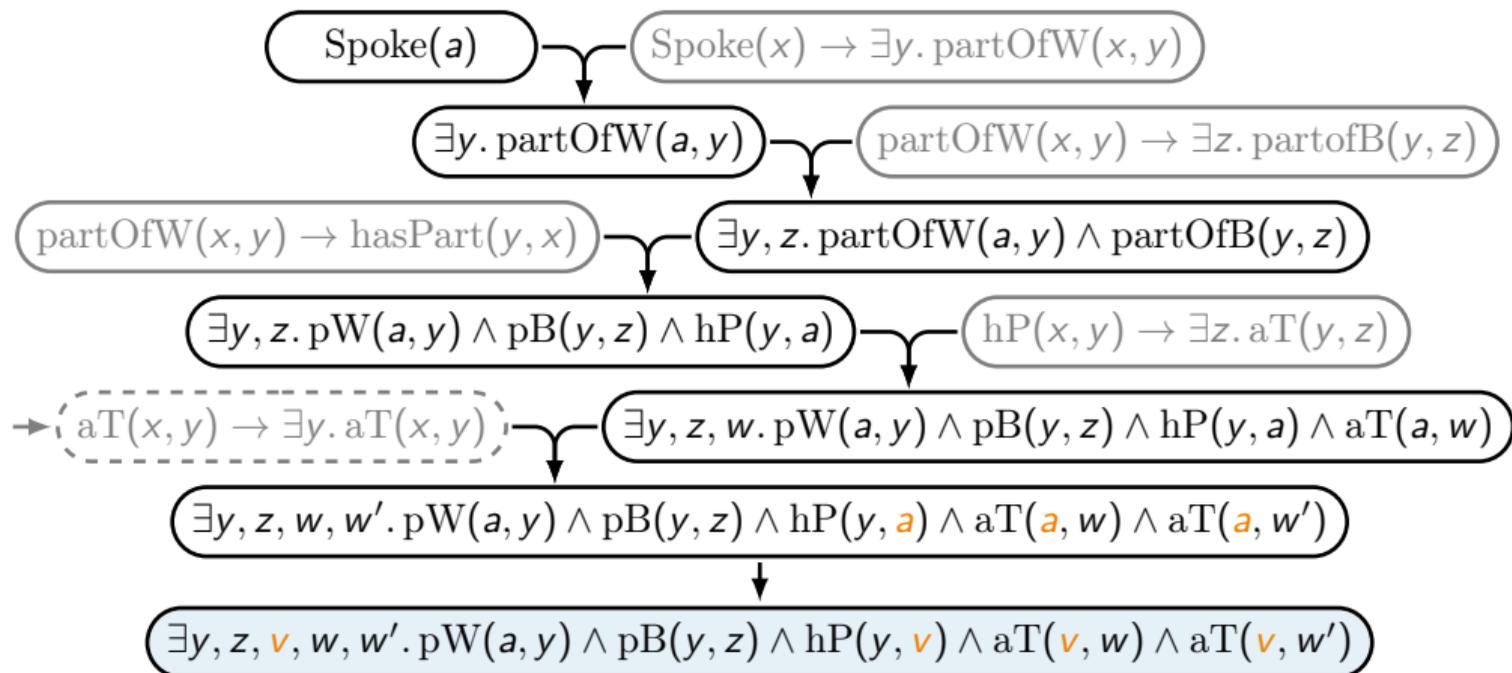
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Deriving Ground Atoms (\mathcal{D}_{sk})

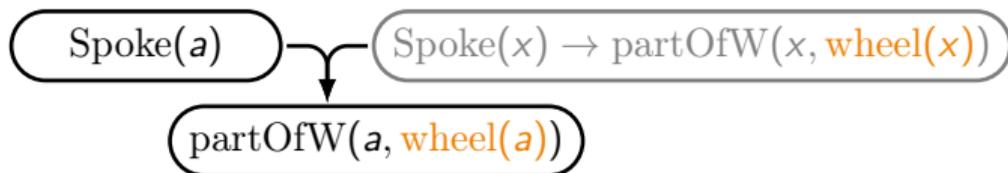
(Borgida, Calvanese, and Rodriguez-Muro 2008)

We can use **Skolem functions** to make proofs more modular and compact.

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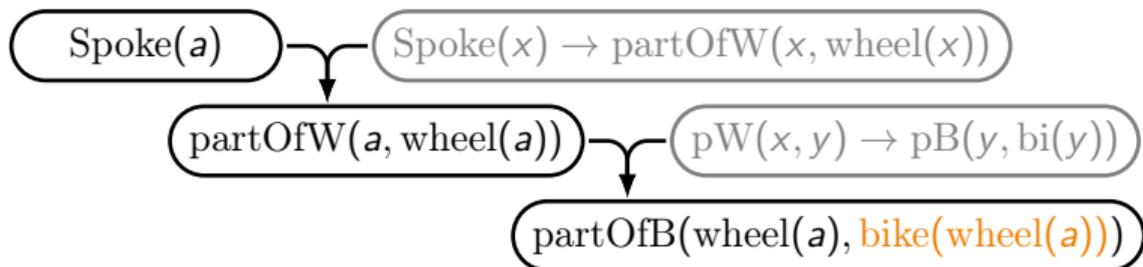
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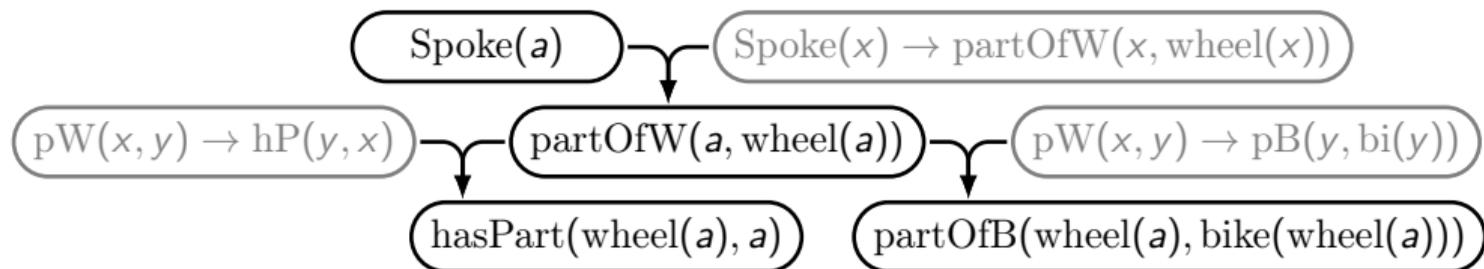
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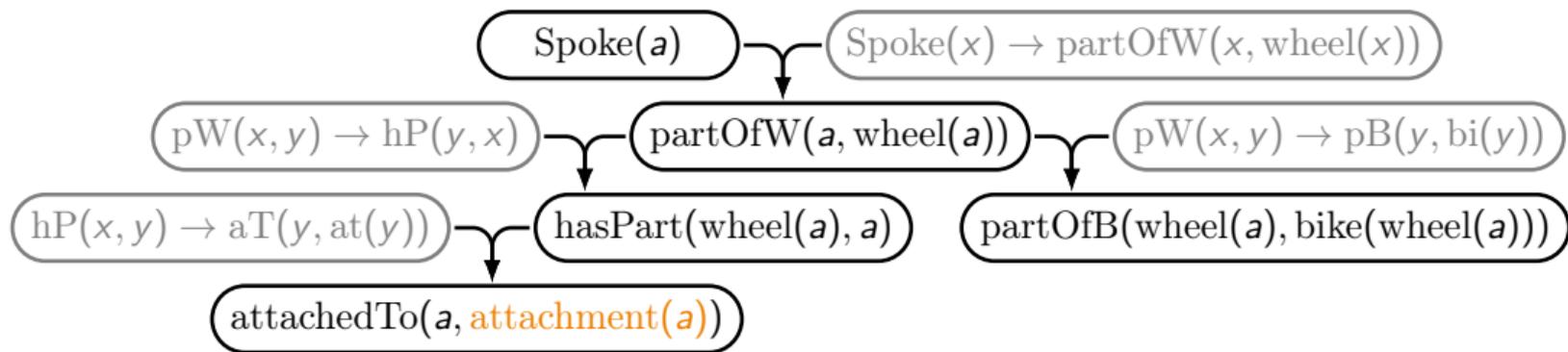
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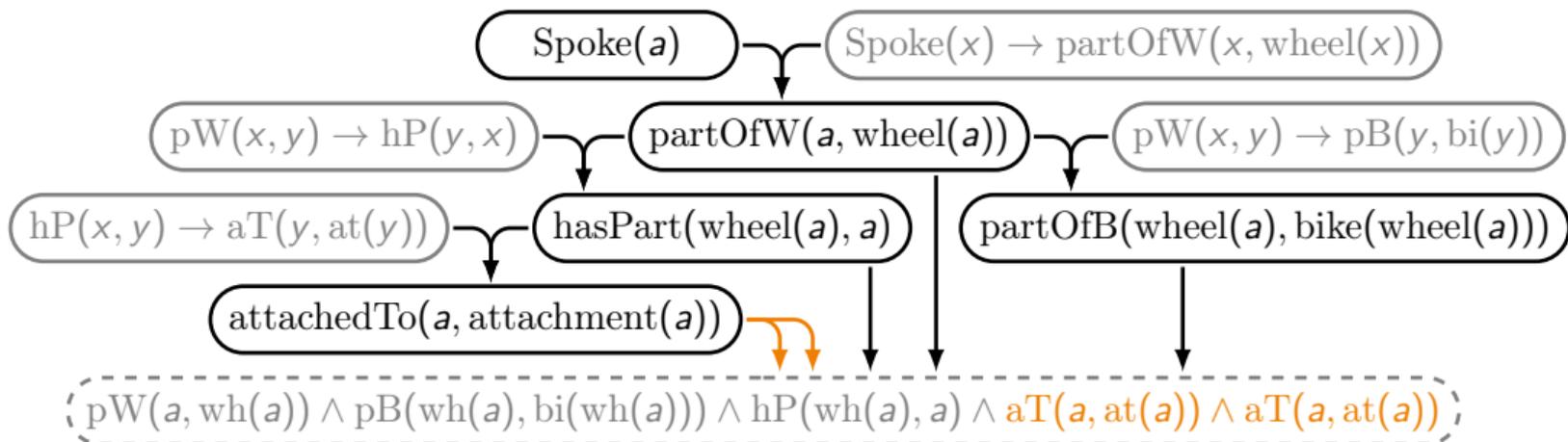
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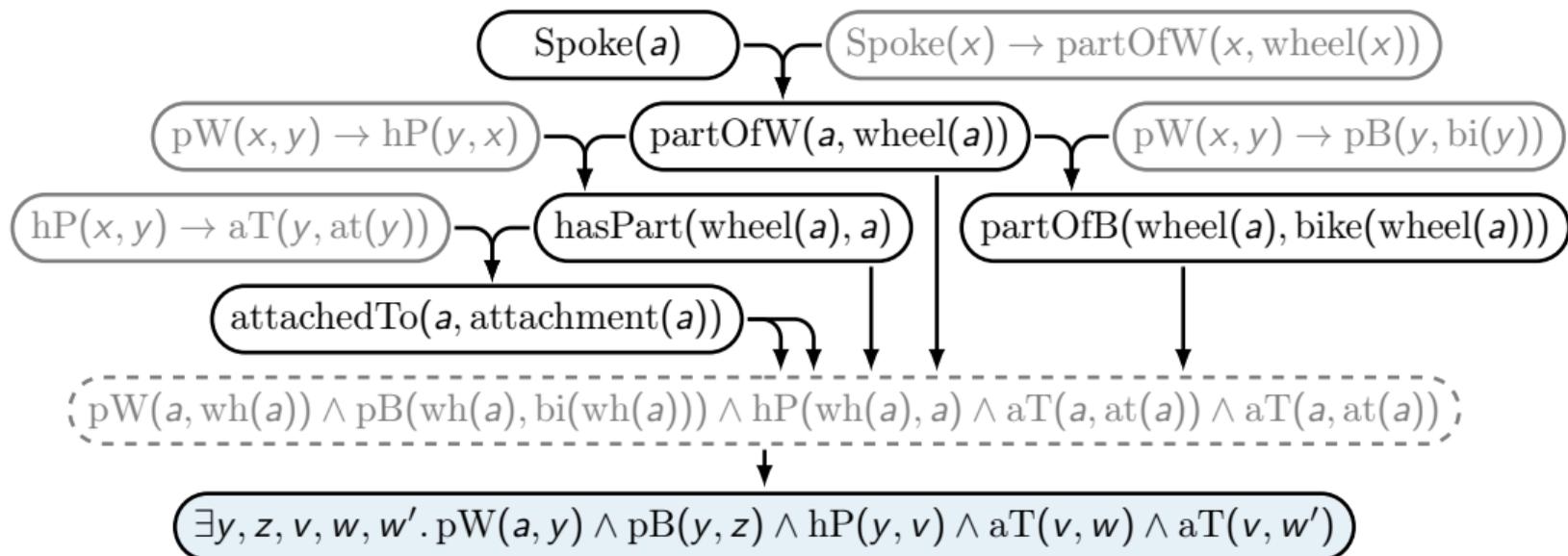
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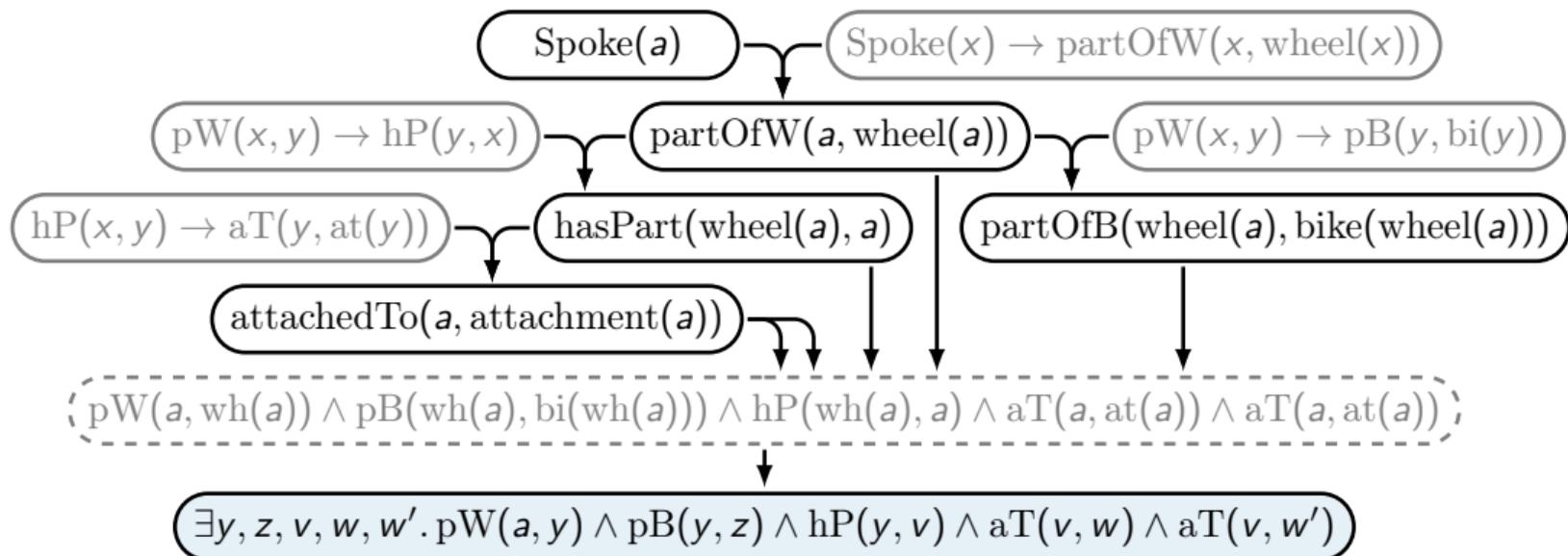
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Only sound w.r.t. Skolemized TBox!

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- Description Logics

- Proofs

- Ontology-Mediated Queries

Proof Systems

- Deriving CQs

- Deriving Ground Atoms

Complexity

Summary

The Complexity of Finding Small (Tree) Proofs

Assuming that $\mathcal{T} \cup \mathcal{A} \models q(a)$ holds, is there a proof of (tree) size $\leq n$ w.r.t. $\mathcal{D}_{cq}/\mathcal{D}_{sk}$?

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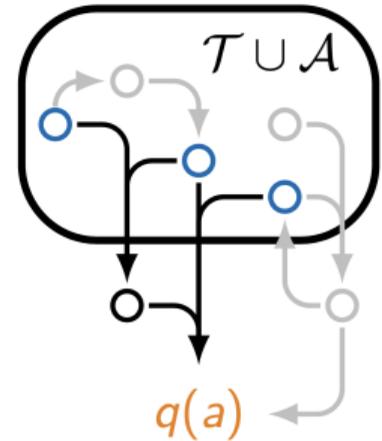
- In AC^0 for $DL-Lite_R$ (exploit query rewritability)

Combined complexity:

- NP-complete for $DL-Lite_R$ (tree size can be bounded by a polynomial)
- In P for **tree-shaped queries** in $DL-Lite_R$ w.r.t. tree size and \mathcal{D}_{sk}
(use placeholders for Skolem terms)
- NP-hard for tree-shaped queries w.r.t. **size or \mathcal{D}_{cq}** and **empty TBox**
(reductions from SAT)

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- Two different kinds of inference rules
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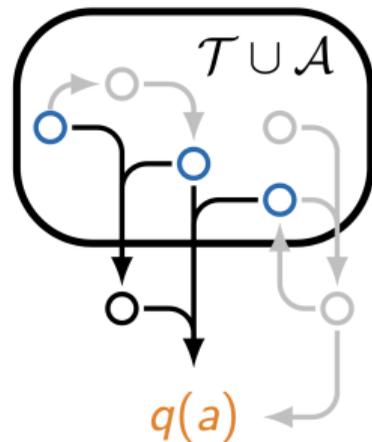


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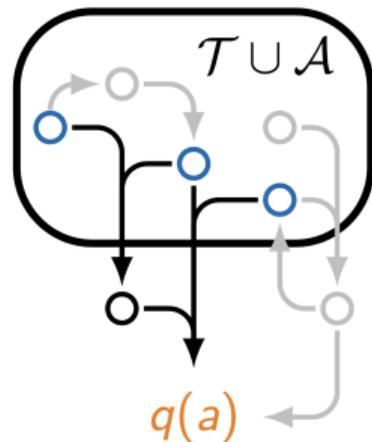


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Thank you!

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