Dresden University of Technology

CENTER FOR PERSPICUOUS COMPUTING

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Technische Universität Dresden

The Shape of $\mathcal{EL}$ Proofs: A Tale of Three Calculi

XLoKR’25, Melbourne, Australia, November 13th, 2025

What are description logics?

Standardized in W3C standard OWL 2 (2012)

Used to describe knowledge in ontologies

SNOMED CT The global language of healthcare

BioPortal

Protégé

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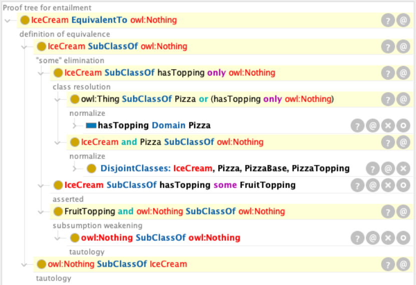
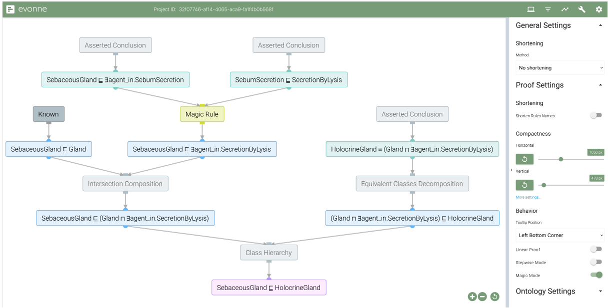
How to explain description logic entailments?



Evee



Evonne

Why is this useful?

$$\left\{\begin{array}{cc} \text{hasPaperAt} \sqsubseteq \text{contributesTo} & \text{KR25Author} \sqsubseteq \exists\text{hasConferencePaperAt.KR25} \\ \exists\text{contributesTo.KR} \sqsubseteq \text{KRCommunityMember} & \text{KR25ShortPaperAuthor} \sqsubseteq \text{KR25Author} \\ \text{hasConferencePaperAt} \sqsubseteq \text{hasPaperAt} & \text{KR25} \sqsubseteq \text{KR} \end{array}\right\}$$

$$\models \quad \text{KR25ShortPaperAuthor} \sqsubseteq \text{KRCommunityMember}$$

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$$
\left\{
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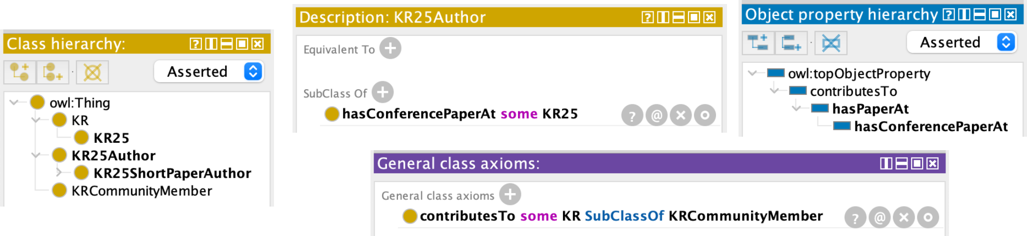

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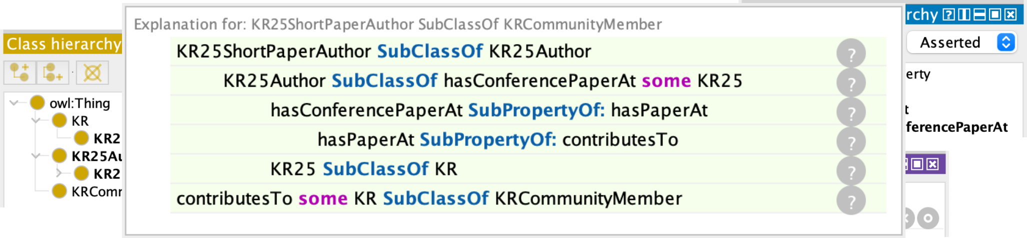

How does reasoning in (some) description logics work?

Pushing the $\mathcal{EL}$ Envelope

Franz Baader and **Sebastian Brandt** and **Carsten Lutz**
Intitute for Theoretical Computer Science
TU Dresden, Germany
lastname@tcs.inf.tu-dresden.de

ENVELOPE ($\mathcal{EL}^{++}$), 2005

CR1 If $C' \in S(C)$, $C' \sqsubseteq D \in \mathcal{C}$, and $D \notin S(C)$
then $S(C) := S(C) \cup \{D\}$

CR4 If $(C, D) \in R(r)$, $D' \in S(D)$, $\exists r.D' \sqsubseteq E \in \mathcal{C}$,
and $E \notin S(C)$
then $S(C) := S(C) \cup \{E\}$

How does reasoning in (some) description logics work?

The Incredible ELK

From Polynomial Procedures to Efficient Reasoning with $\mathcal{EL}$ Ontologies

Yevgeny Kazakov · Markus Krötzsch · František Simančík

ENVELOPE ($\mathcal{EL}^{++}$), 2005

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ELK ($\mathcal{EL}_\bot^+$), 2014

$$\mathsf{R}_\sqsubseteq \ \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$$

$$\mathsf{R}_\exists^+ \ \frac{E \xrightarrow{R} C \quad C \sqsubseteq D}{E \sqsubseteq \exists S.D} : \begin{array}{l} R \sqsubseteq_\mathcal{O}^* S \\ \exists S.D \text{ occurs negatively in } \mathcal{O} \end{array}$$

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TEXTBOOK ($\mathcal{EL}$), 2017

$$\mathsf{CR3}\ \frac{A_1 \sqsubseteq A_2 \quad A_2 \sqsubseteq A_3}{A_1 \sqsubseteq A_3}$$

$$\mathsf{CR5}\ \frac{A \sqsubseteq \exists r.A_1 \quad A_1 \sqsubseteq B_1 \quad \exists r.B_1 \sqsubseteq B}{A \sqsubseteq B}$$

How does reasoning in (some) description logics work?

To compare them, we use $\mathcal{ELH}_\bot$

ENVELOPE ($\mathcal{EL}^{++}$), 2005

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ELK ($\mathcal{EL}^+_\bot$), 2014

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What do the proofs look like?

$$\dfrac{\dfrac{\mathsf{KR25SPA} \sqsubseteq \mathsf{KR25A} \qquad \mathsf{KR25A} \sqsubseteq \exists \mathsf{hasCP}.\mathsf{KR25}}{\mathsf{KR25SPA} \sqsubseteq \exists \mathsf{hasCP}.\mathsf{KR25}} \qquad \mathsf{KR25} \sqsubseteq \mathsf{KR} \qquad \dfrac{\mathsf{hasCP} \sqsubseteq \mathsf{hasP} \qquad \mathsf{hasP} \sqsubseteq \mathsf{cT}}{\mathsf{hasCP} \sqsubseteq \mathsf{cT}} \qquad \exists \mathsf{cT}.\mathsf{KR} \sqsubseteq \mathsf{KRCM}}{\mathsf{KR25SPA} \sqsubseteq \mathsf{KRCM}}$$

TEXTBOOK

What do the proofs look like?

TEXTBOOK

$$\dfrac{\dfrac{\mathrm{KR25SPA} \sqsubseteq \mathrm{KR25A} \qquad \mathrm{KR25A} \sqsubseteq \exists \mathrm{hasCP.KR25}}{\mathrm{KR25SPA} \sqsubseteq \exists \mathrm{hasCP.KR25}} \qquad \mathrm{KR25} \sqsubseteq \mathrm{KR} \qquad \dfrac{\mathrm{hasCP} \sqsubseteq \mathrm{hasP} \qquad \mathrm{hasP} \sqsubseteq \mathrm{cT}}{\mathrm{hasCP} \sqsubseteq \mathrm{cT}} \qquad \exists \mathrm{cT.KR} \sqsubseteq \mathrm{KRCM}}{\mathrm{KR25SPA} \sqsubseteq \mathrm{KRCM}}$$

ELK

$$\dfrac{\dfrac{\dfrac{\mathrm{KR25SPA} \sqsubseteq \mathrm{KR25A} \qquad \mathrm{KR25A} \sqsubseteq \exists \mathrm{hasCP.KR25}}{\mathrm{KR25SPA} \sqsubseteq \exists \mathrm{hasCP.KR25}} \qquad \mathrm{KR25} \sqsubseteq \mathrm{KR} \qquad \dfrac{\mathrm{hasCP} \sqsubseteq \mathrm{hasP} \qquad \mathrm{hasP} \sqsubseteq \mathrm{cT}}{\mathrm{hasCP} \sqsubseteq \mathrm{cT}}}{\mathrm{KR25SPA} \sqsubseteq \exists \mathrm{cT.KR}} \qquad \exists \mathrm{cT.KR} \sqsubseteq \mathrm{KRCM}}{\mathrm{KR25SPA} \sqsubseteq \mathrm{KRCM}}$$

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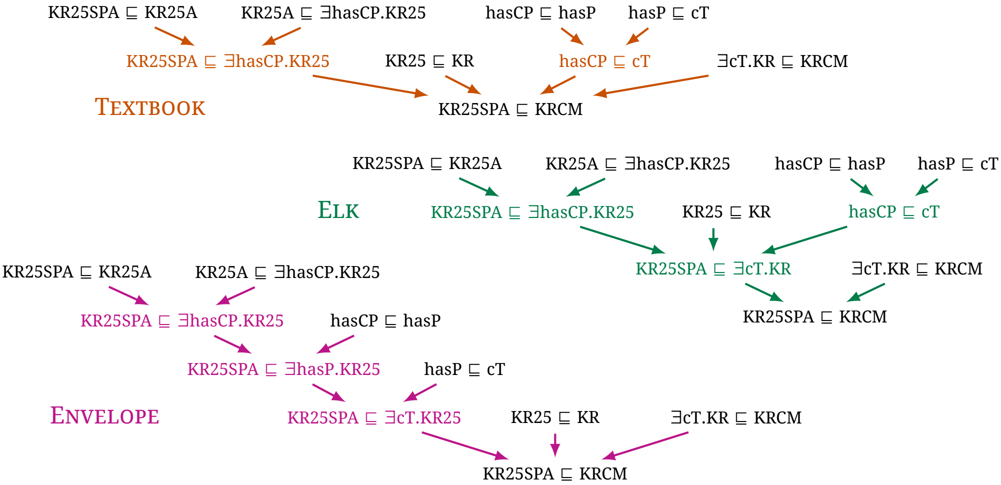
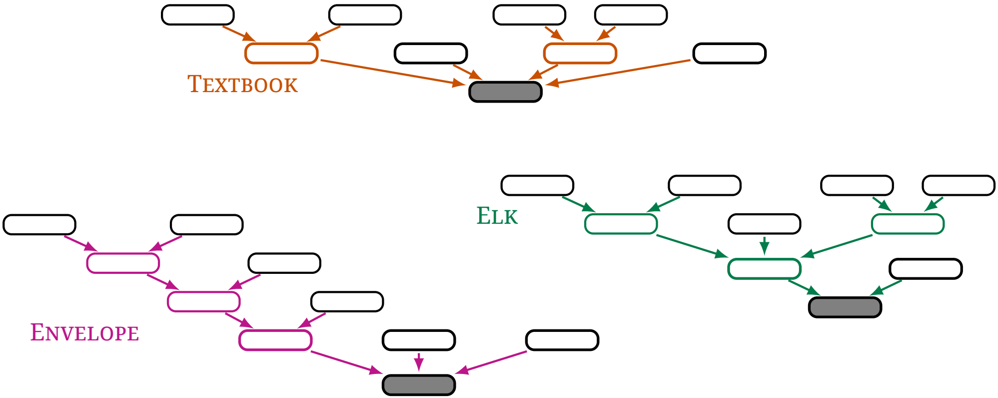


What is the structure of the proofs?



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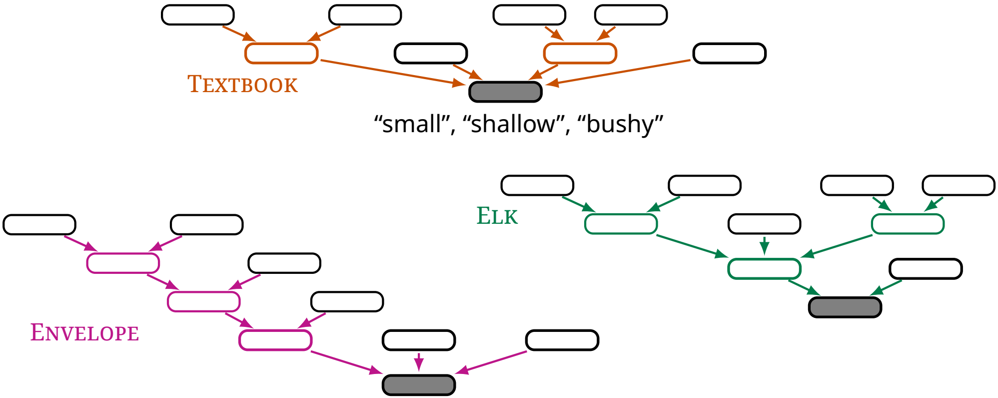

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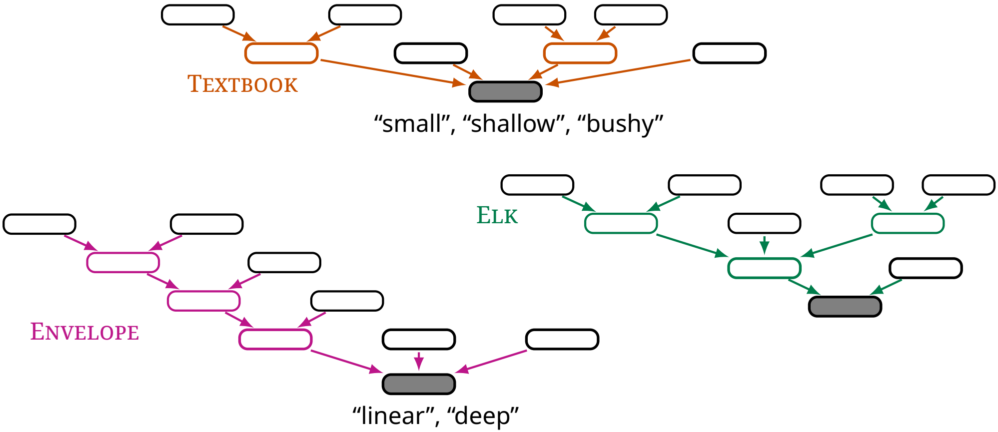
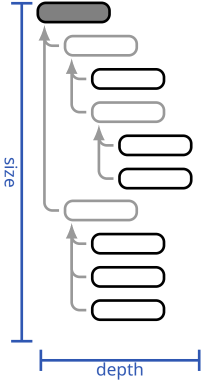


How can we display and measure proofs?



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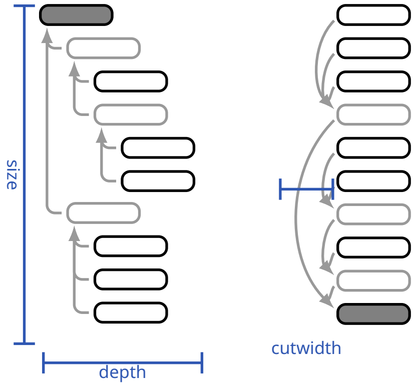

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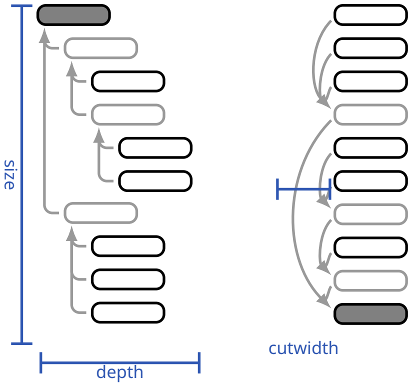
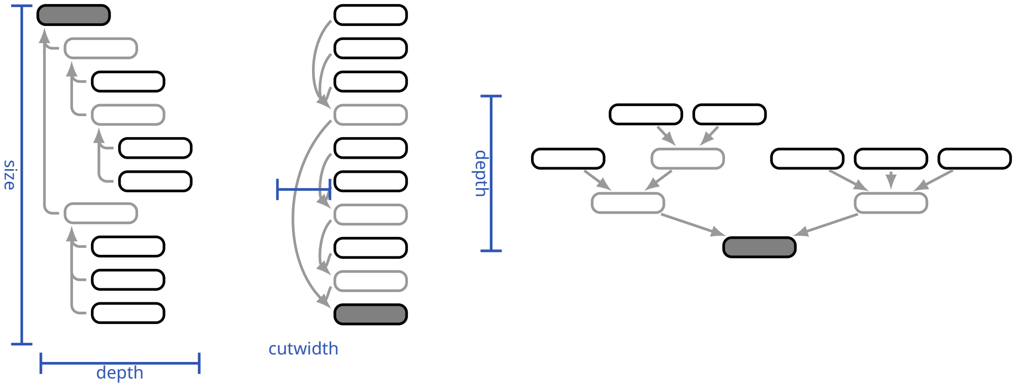


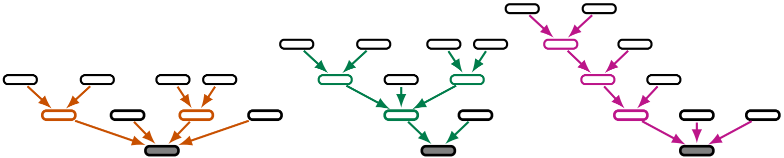
- cutwidth is NP-complete for arbitrary graphs (Bodlaender, Fellows, and Thilikos 2009)
- directed cutwidth on trees can be computed in polynomial time (Theorem 1)

How can we display and measure proofs?



- bushiness score: $\frac{\text{size}}{\text{depth+1}}$ ("average number of nodes per level")
- average step complexity (Horridge, Bail, Parsia, and Sattler 2013)

What does this mean for our example?



	Textbook	Elk	Envelope	
size	**9**	10	10	
depth	**2**	3	4	
cutwidth	4	**3**	**3**	
bushiness	3	2.5	**2**	
complexity	230	**215**	257.5	/ 410

How can we generate the proofs?

$$\left\{ \begin{array}{cc} \text{hasPaperAt} \sqsubseteq \text{contributesTo} & \text{KR25Author} \sqsubseteq \exists\text{hasConferencePaperAt.KR25} \\ \exists\text{contributesTo.KR} \sqsubseteq \text{KRCommunityMember} & \text{KR25ShortPaperAuthor} \sqsubseteq \text{KR25Author} \\ \text{hasConferencePaperAt} \sqsubseteq \text{hasPaperAt} & \text{KR25} \sqsubseteq \text{KR} \end{array} \right\}$$

$$\Downarrow$$

RDF triples:

```
:hasPaperAt rdf:type owl:ObjectProperty .
:hasPaperAt rdfs:subPropertyOf :contributesTo .

:KR25Author rdf:type owl:Class .
:KR25Author rdfs:subClassOf _:1 .
_:1 rdf:type owl:Restriction .
_:1 owl:onProperty :hasConferencePaperAt .
_:1 owl:someValuesFrom :KR25 .

...
```

How can we generate the proofs?

$$\mathsf{R}_{\sqsubseteq}\ \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$$

$$\mathsf{R}_{\exists}^{+}\ \frac{E \xrightarrow{R} C \quad C \sqsubseteq D}{E \sqsubseteq \exists S.D} : \begin{array}{l} R \sqsubseteq_{\mathcal{O}}^{*} S \\ \exists S.D \text{ occurs negatively in } \mathcal{O} \end{array}$$

$$\Downarrow$$

Existential rules:

```
...
inf:subClassOf(?C,?E) :- inf:subClassOf(?C,?D), nf:subClassOf(?D,?E) .
inf:subClassOf(?E,?Y) :- inf:ex(?E,?R,?C), inf:subClassOf(?C,?D), nf:subProp(?R,?S),
                         nf:exists(?Y,?S,?D), nf:isSubClass(?Y) .
...
```

(Ivliev, Gerlach, Meusel, Steinberg, and Krötzsch 2024)

How can we generate the proofs?

RDF triples
Existential rules

⟹

Nemo
Graph Rule Engine

(Ivliev, Gerlach, Meusel, Steinberg, and Krötzsch 2024)

⟹

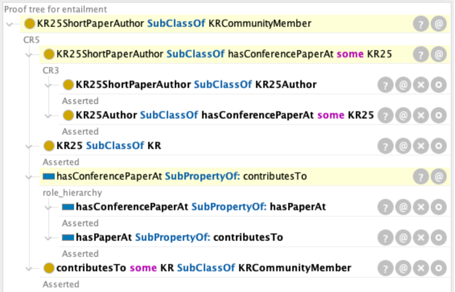
Trace

How can we generate the proofs?

Trace

$\Longrightarrow$

DL Proof



Which proofs are better?

1573 prototypical $\mathcal{ELH}_\bot$ reasoning tasks from OWL Reasoner Evaluation 2015

(Parsia, Matentzoglu, Gonçalves, Glimm, and Steigmiller 2017)

justifications of average size 10.7 (max. 25), average complexity 495 (max. 1470)

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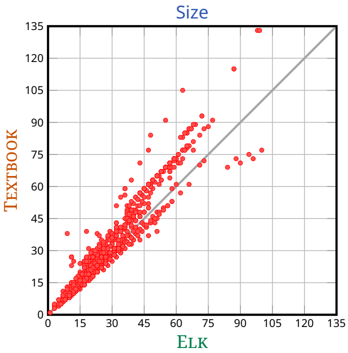

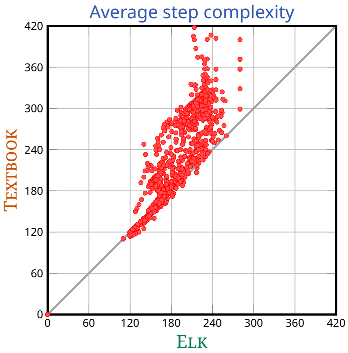

Elk proofs are smaller and easier to understand

Which proofs are better?

1573 prototypical $\mathcal{ELH}_\bot$ reasoning tasks from OWL Reasoner Evaluation 2015

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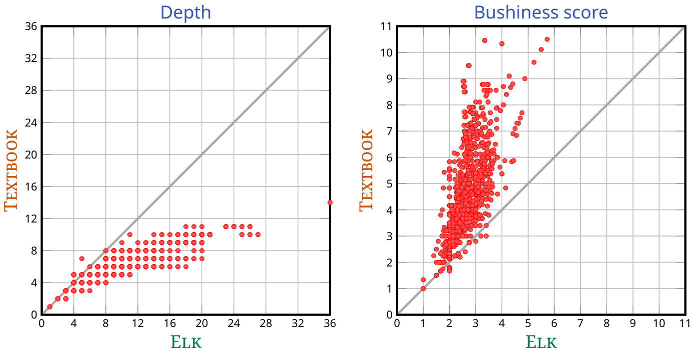
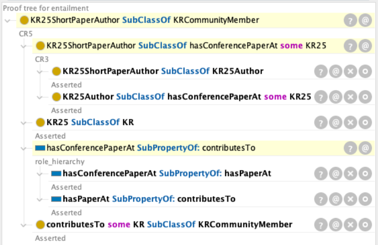
justifications of average size 10.7 (max. 25), average complexity 495 (max. 1470)



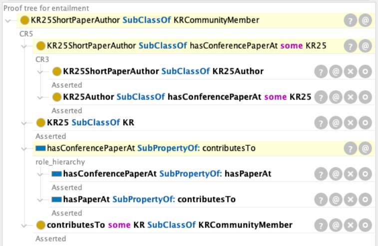
TEXTBOOK proofs are better for tree-like views

How can you use this right now?



github.com/de-tu-dresden-inf-lat/evee

How can you use this right now?



`github.com/de-tu-dresden-inf-lat/evee`

`github.com/knowsys/nemo`

Nemo
Graph Rule Engine

What comes next?

- Generate all traces with NEMO, not just one
- Compare proofs from ELK calculus to proofs from ELK reasoner

  (Kazakov and Klinov 2014; Kazakov, Klinov, and Stupnikov 2017)
- Calculi for more expressive (Horn and non-Horn) logics

Thank you!

References I

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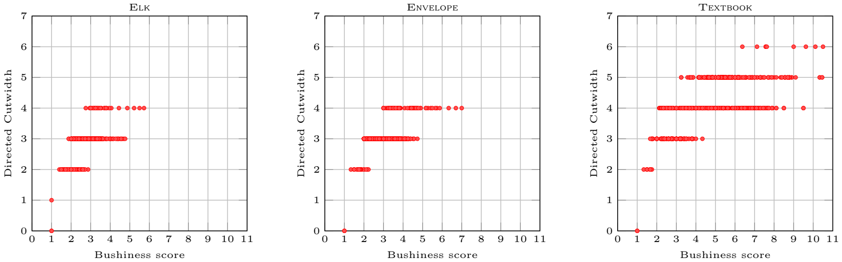
Kazakov, Yevgeny, Markus Krötzsch, and František Simančík (2014). "The Incredible ELK: From Polynomial Procedures to Efficient Reasoning with $\mathcal{EL}$ Ontologies". In: *J. Autom. Reason.* 53.1. DOI: 10.1007/s10817-013-9296-3.

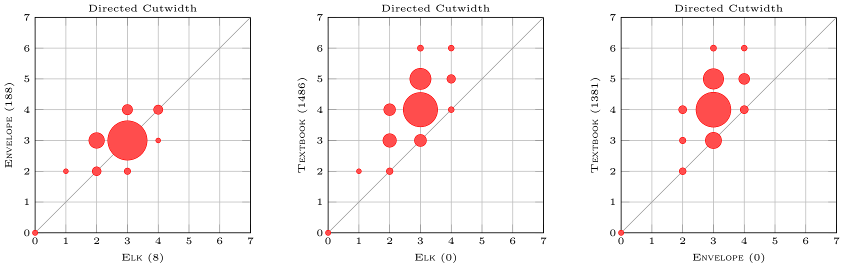
References II

Parsia, Bijan, Nicolas Matentzoglu, Rafael S. Gonçalves, Birte Glimm, and Andreas Steigmiller (2017). "The OWL Reasoner Evaluation (ORE) 2015 Competition Report". In: *J. Autom. Reason.* 59.4. DOI: 10.1007/S10817-017-9406-8.

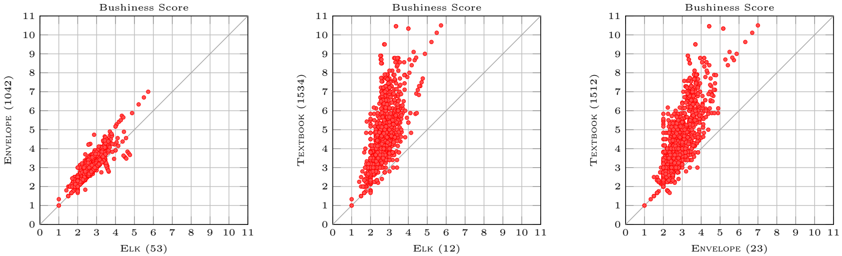
Relation between cutwidth and bushiness



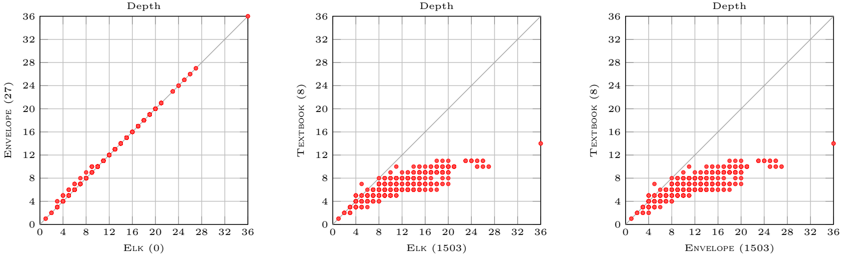
Directed cutwidth



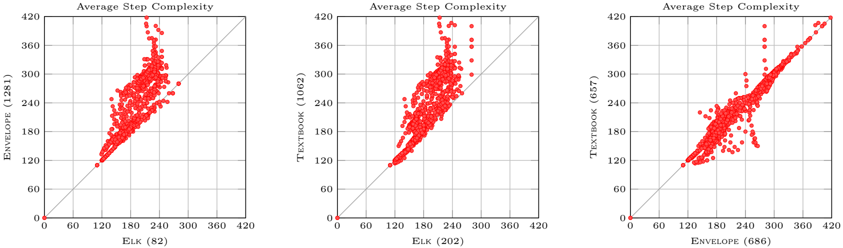
Bushiness score



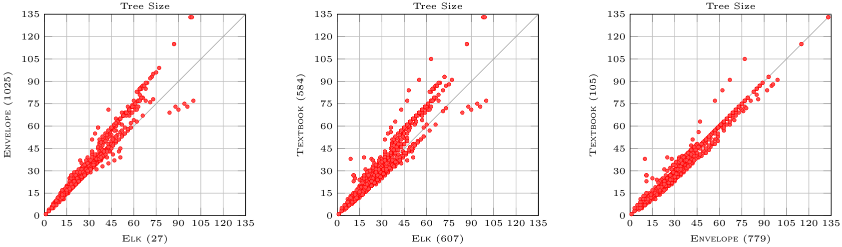
Depth


Depth
Envelope (27)
Elk (0)
Depth
Textbook (8)
Elk (1503)
Depth
Textbook (8)
Envelope (1503)
0
4
8
12
16
20
24
28
32
36

Average step complexity



Tree size



Tree Size
Envelope (1025)
Elk (27)
Tree Size
Textbook (584)
Elk (607)
Tree Size
Textbook (105)
Envelope (779)
0
15
30
45
60
75
90
105
120
135