INTRODUCTION TO NONMONOTONIC REASONING

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Section 3

Default Logic

Subsection 3.1

Introducing defaults and default logics
Suppose you are asked how you get to the university in the morning.

By bike (usually)! \( \text{gotoWork : byBike} \)

\[ \text{byBike} \]

**new information:** It is snowing heavily and your bike’s tire is flat.
(You cannot assume that you go by bike.)

The default is no longer applicable \( \rightsquigarrow \) Revise previous conclusion

Why can **classical logic** not model this?

\[ \text{goToUni} \land \neg \text{snow} \land \neg \text{FlatTire} \rightarrow \text{useBike} \]

- There are more reasons not to use the bike, e.g. broke brakes, demonstration in the city, . . . The formula would need to **list all possible obstacles**!
- **All preconditions** would need to be established to be true, so that the rule applies!
Default reasoning for kinds of common sense reasoning

Defaults can be used to model several forms of (common sense) reasoning.

- **Prototypical reasoning**: most instances of a category have a property.
  
  “Typically, children have parents.”
  
  \[ \text{child}(X) : \text{hasParents}(X) \]
  
  \[ \text{hasParents}(X) \]

- **No-risk reasoning**: concerns situations where a conclusion is drawn even if it is not the most probable, because another conclusion could lead to disaster.
  
  “In absence of evidence to the contrary assume the accused is innocent.”
  
  \[ \text{accused}(X) : \text{innocent}(X) \]
  
  \[ \text{innocent}(X) \]

- **Best-guess reasoning**: for instance, we know that there are some shopping centers in this city and some are open on Sundays, but we don’t know which one. We would try the closest first, although we don’t have evidence of it.
  
  \[ \text{closest}(X) : \text{openSundays}(X) \]
  
  \[ \text{openSundays}(X) \]

Default reasoning appears in many application domains: legal reasoning, diagnosis, reasoning about actions, etc.
Introducing defaults and default logics

- **Default logics** were introduced by Ray Reiter in 1980.
- **Default reasoning** appears when reasoning is done under the **closed world assumption** and using **inference rules that admit exceptions** (rules that hold under the **normality assumption**)
  
  “... in absence of any information to the contrary, assume ...”

- **Classical inference rules** sanction the derivation of a formula whenever some other formulae are derived.
- **Default rules** require an **additional consistency condition** to hold.

E.g.: the rule “normally birds fly” is represented as $\frac{\text{bird}(x) : \text{flies}(x)}{\text{flies}(x)}$

This states that:

“if $\text{bird}(J)$ is given or derived for a particular ground term $J$ and $\text{flies}(J)$ is consistent (there is no information that $\neg \text{flies}(J)$ holds), then $\text{flies}(J)$ can be derived “by default”.

Consistent with what? Set of formulae that can “reasonably” be accepted based on the available information.
Syntax of Default Logic

Definition 3.1 (Default theory)
A default theory is a pair \((W, D)\) consisting of
- \(W\): a set of FOL formulae (called facts or axioms) and
- \(D\): a countable set of defaults

A default \(\delta\) has the form
\[
\varphi : \psi_1, \ldots, \psi_n \\
\chi
\]
where \(\varphi, \psi_1, \ldots, \psi_n, \) and \(\chi\) are closed FOL formulae and \(n > 0\).

The formula
- \(\varphi\) is called the prerequisite (denoted by \(pre(\delta)\)),
- \(\psi_1, \ldots, \psi_n\) the justifications (denoted by \(just(\delta)\)), and
- \(\chi\) the consequent of \(\delta\) (denoted by \(cons(\delta)\)).
Why closed formulae?

What happens when open formulae are used?

Actually,

\[
\text{bird}(x) : \text{flies}(x) \\
\text{flies}(x)
\]

is not a default according to Definition 3.1, since the formulae are not closed. We call such “defaults” open defaults. An open default is interpreted as a default schema representing a (possibly infinite) set of defaults.

A default schema differs from a default in that \(\varphi, \psi_1, \ldots, \psi_n, \chi\) are arbitrary FOL formulae (may contain free variables). A default schema defines a set of defaults

\[
\varphi\sigma : \psi_1\sigma, \ldots, \psi_n\sigma \\
\chi\sigma
\]

for all ground substitutions \(\sigma\) that assign values to all free variables occurring in the schema.

\(\sim\) Free variables : interpreted as being universally quantified over the whole default schema.
Why closed formulae?

Trying to use quantifiers

The open default

\[
\text{bird}(x) : \text{flies}(x) \\
\hline \\
\text{flies}(x)
\]

would read under

- universally quantified variables as
  
  “If all X are birds, and if for all X we may assume that they fly, 
  then we conclude that all X fly.”
  
  - Does not match the intuition. All birds are treated uniformly.
  - Would only be applicable, if every object in the domain is a bird.

- existentially quantified variables as
  
  “If there is a bird and if there is an X that flies, then conclude that 
  there is some flying object.”
  
  - Would not allow to conclude from bird(tweety) that 
    flies(tweety) holds.
  - instead we would conclude: \( \exists X \text{ flies}(X) \)

\[\sim\] Using one of the quantifiers is not adequate!
Towards the semantics of defaults

The informal meaning of a default $\varphi : \psi_1, \ldots, \psi_n$ is:

“If $\varphi$ is known, and if it is consistent to assume that $\psi_1, \ldots, \psi_n$, then conclude $\chi$.”

In the formal semantics we must say

1. where $\varphi$ should be included
2. with what should $\psi_1, \ldots, \psi_n$ be consistent

With what should $\psi_1, \ldots, \psi_n$ be consistent? A first attempt: the facts.

Consider the default

$$
friend(X, Y) \land friend(Y, Z) : friend(X, Z) : friend(X, Z)
$$

Given the facts: $friend(tom, bob), friend(bob, sally), friend(sally, tina)$. Wanted conclusion: $friend(tom, tina)$

This is only possible if we apply the appropriate instance of the default schema to $friend(sally, tina)$ and $friend(tom, sally)$. But $friend(tom, sally)$ is derived by a previous application of the default schema!

Without this intermediate result and from the facts alone, we could not derive this.
Example 3.2
Let’s consider $T = (W, D)$ with $W = \{\text{green, ADACmember}\}$ and $D = \{\delta_1, \delta_2\}$, where

$$
\delta_1 = \frac{\text{green : } \neg \text{likesCars}}{
eg \text{likesCars}} \quad \text{and} \quad \delta_2 = \frac{\text{ADACmember : } \text{likesCars}}{\text{likesCars}}
$$

If consistency is tested against $W$, both defaults can be applied. Deriving $\neg \text{likesCars}$ and $\text{likesCars}$, which is a contradiction!

Alternative:
apply the first default $\delta_1$, check for consistency with the knowledge derived so far. Would block the application of the second default $\delta_2$. 
Informal semantics of defaults

A general formulation:
If $\varphi$ is currently known, and if all $\psi_i$ are consistent with the current knowledge base, then conclude $\chi$.

The current knowledge base $E$ is obtained from
- the facts and
- the consequents of some defaults that have been applied previously.

A more formal version:
Default $\delta = \frac{\varphi}{\psi_1, \ldots, \psi_n} \chi$ is applicable to a deductively closed set of formulae $E$ iff $\varphi \in E$ and $\neg \psi_1 \notin E, \ldots, \neg \psi_1 \notin E$. 
Towards extensions

Example 3.2 suggests that there can be several competing knowledge bases which maybe inconsistent with each other.

Extensions represent possible world views which are based on the given default theories. They seek to extend the set of known facts with “reasonable” conjectures based on the available defaults.

Desirable properties of extensions:

- an extension should include the set $W$ of facts—the certain information
- an extension should be deductively closed (Keep classical reasoning! Derive more from the defaults.)
- an extension should be closed under the application of the defaults. Apply defaults exhaustively.

Formally: if $\varphi, \psi_1, \ldots, \psi_n \in D$, $\varphi \in E$ and $\neg \psi_1 \not\in E$, $\ldots$, $\neg \psi_n \not\in E$ then $\chi \in E$.

$\Rightarrow$ Extensions are maximal possible world views.
Towards extensions — unwanted effects

1. "Ungrounded" beliefs
An extension must not contain "ungrounded" beliefs, i.e. every formula in the extension must be derivable from $W$ and the consequents of applied defaults. We require extensions to be minimal w.r.t. to these properties.

Consider: $T = (W, D)$ with $W = \{ \text{german} \}$ and $D = \{ \frac{\text{german}}{\text{drinksBeer}} \}\{\frac{\text{drinksBeer}}{-\text{drinksBeer}}\}$

Now, $E = \text{Th}(\{\text{german}, -\text{drinksBeer}\})$ is minimal w.r.t. to the properties, but unintuitive.

2. Applications of defaults can contradict the application of an earlier default.
Consider:

<table>
<thead>
<tr>
<th>true : $\text{creditworthy}$</th>
<th>true : $-\text{creditworthy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>approveCredit</td>
<td>$-\text{creditworthy}$</td>
</tr>
</tbody>
</table>

We apply the first default, since nothing contradicts the assumption $\text{creditworthy}$. We then apply the second, since $-\text{creditworthy}$ is consistent with the knowledge, $-\text{creditworthy}$ is derived.

Inclusion of $-\text{creditworthy}$ shows a-posteriori that, we should not have assumed $\text{creditworthy}$.
Subsection 3.2
Operational semantics of Default Logic

• based on the process in which inferences are drawn
• gives a procedure that can be applied

Idea:
• apply defaults as long as possible
• If a default should not have been applied, backtrack and try an alternative
Operational Semantics

Given a default theory $T = (W, D)$ let $\Pi = (\delta_0, \delta_1 \ldots)$ be (a finite or infinite) sequence of defaults from $D$ without multiple occurrences. (Possible order in which some defaults from $D$ are applied.)

$\Pi[k]$ denotes the initial segment of sequence $\Pi$ of length $k$.\footnote{We assume (from now on) that the length of $\Pi$ is at least $k$.}

Each sequence $\Pi$ is associated with two sets: $In(\Pi)$ and $Out(\Pi)$

- $In(\Pi) = Th(W \cup \{ \text{cons}(\delta) \mid \delta \text{ occurs in } \Pi \})$.
- $Out(\Pi) = \{ \neg \psi \mid \psi \in \text{just}(\delta) \text{ for some } \delta \text{ in } \Pi \}$.

Intuition:

- $In(\Pi)$ represents the current knowledge base after the defaults in $\Pi$ have been applied
- $Out(\Pi)$ represents the formulae that should not become true even after subsequent application of other defaults.
Example: default sequences

Example 3.3
Consider $T = (W, D)$ with $W = \{a\}$ and the defaults from $D$:

$$\delta_1 = \frac{a \cdot \neg b}{\neg b}, \quad \delta_2 = \frac{b \cdot c}{c}$$

For $\Pi_a = (\delta_1)$ we have $ln(\Pi_a) = Th(\{a, \neg b\})$ and $Out(\Pi_a) = \{b\}$.
For $\Pi_b = (\delta_2, \delta_1)$ we have $ln(\Pi_b) = Th(\{a, c, \neg b\})$ and $Out(\Pi_b) = \{\neg c, b\}$

We have not assured that the defaults can be applied in the order given. 
$(\delta_2, \delta_1)$ cannot be applied in this order, since $b \not\in ln(\emptyset) = Th(W) = Th(a)$.

"Applicable sequences" are formalized by the notion of a process.
Processes

Definition 3.4 (Process, successful, closed)
\( \Pi \) is a process of \( T \) iff \( \delta_k \) is applicable to \( \text{ln}(\Pi[k]) \) for every \( k \) s.t.\(^2 \) \( \delta_k \) occurs in \( \Pi \).

Let \( \Pi \) be a process. We define:

- \( \Pi \) is successful iff \( \text{ln}(\Pi) \cap \text{out}(\Pi) = \emptyset \). Otherwise, it is failed.
- \( \Pi \) is closed iff every \( \delta \in D \) that is applicable to \( \text{ln}(\Pi) \) already occurs in \( \Pi \).

Intuition:
Success of a process captures that it is "okay" to have assumed the justifications of the applied defaults; no formula \( \neg \psi \in \text{out}(\Pi) \) is part of the current knowledge base, so it was consistent to assume \( \psi \).

Closed processes correspond to the extension being closed under application of the defaults.
Example: properties of processes

Consider the default theory \( T = (W, D) \) with \( W = \{a\} \) and \( D \) containing

\[
\delta_1 = \frac{a : \neg b}{d}, \quad \delta_2 = \frac{\text{true} : c}{b}
\]

\( \Pi_1 = (\delta_1) \) is successful, but not closed, since \( \delta_2 \) may be applied to \( \text{In}(\Pi_1) = \text{Th}(\{a, d\}) \).

\( \Pi_2 = (\delta_2, \delta_1) \) is closed, but not successful. Since both \( \text{In}(\Pi_2) = \text{Th}(a, b, d) \) and \( \text{Out}(\Pi_2) = \{b, \neg c\} \) contain \( b \).

\( \Pi_2 = (\delta_2) \) is a closed and successful process of \( T \).
Definition 3.5 (Extension)
Let $T$ be a default theory. A set of formulae $E$ is an extension of $T$ iff there is some closed and successful process $\Pi$ s.t. $E = \text{In}(\Pi)$.

This definition may be applied directly to concrete examples.
To find a successful process, it suffices to generate a process $\Pi$, test whether $\text{In}(\Pi) \cap \text{Out}(\Pi) = \emptyset$ holds. If not, then backtrack.

A (in)finite default theory is a default theory, where $D$ has (in)finitely many elements.
For finite default theories ensuring closure is conceptually easy: apply an applicable default that has not been applied yet, until no more a left.

How about closure of infinite default theories?
Closure of infinite theories

Lemma 3.6

An infinite process $\Pi$ of a default theory $T = (W, D)$ is closed iff each default in $D$ that is applicable to $\text{ln}(\Pi[k])$, for infinitely many numbers $k$, is already contained in process $\Pi$.

The proof of Lemma 3.6 needs:

Compactness Theorem

The compactness theorem says that a set of $M$ of formulae is satisfiable iff every finite subset of $M$ is satisfiable.

“A set of first-order sentences has a model iff every finite subset of it has a model. “

method for constructing models of any set of sentences that is finitely consistent.

Proof of Lemma 3.6: blackboard

A strategy to guarantee the closure of an infinite process $\Pi$ must ensure that any default which from $k$ on, demands application, will eventually be applied.

(A.k.a. fairness condition from concurrent programming)
Finding closed and successful processes

The process of finding a closed and successful process can be represented by a kind of (search) tree.

**Definition 3.7 (Process tree)**
Let $T = (W, D)$ be a default theory. A process tree is tree $G = (V, E)$, s.t. all nodes $v \in V$ are labeled with two sets of formulae:

- an In-set $In(v)$ and
- an Out-set $Out(v)$.

The root of $G$ is labeled with $Th(W)$ as In-set and $\emptyset$ as Out-set.

The edges in $E$ correspond to default applications and are labelled with the default applied.

The paths of a process are the paths in $G$ starting at the root.

A node $v$ is expanded if $In(v) \cap Out(v) = \emptyset$, otherwise it is marked "failed" and is a leaf of the process tree.
Definition 3.7: Process tree (cont.)

If \( v \in V \) is expanded it possesses for each default \( \delta = \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi} \) one successor node \( w \) that

- does not appear on the path from the root node to \( v \),
- is applicable to \( \text{In}(v) \),
- is connected to \( v \) by an edge labeled with \( \delta \), and
- is labeled with \( \text{Th} (\text{In}(v) \cup \{\chi\}) \) and \( \text{Out}(v) \cup \{\neg \psi_1, \ldots, \neg \psi_n\} \).
Subsection 3.3

Original semantics of default logics

- original definition by Ray Reiter
- fixed-point based, not constructive
Consistency w.r.t. to what?

When applying defaults we need to ensure consistency. But consistency w.r.t. to which theory?

We consider again Example 3.2:

\[ T = (W, D) \text{ with } W = \{ \text{green, ADACmember} \} \text{ and } D = \{ \delta_1, \delta_2 \}, \text{ where} \]

\[ \delta_1 = \frac{\text{green : } \neg \text{likesCars}}{\neg \text{likesCars}} \text{ and } \delta_2 = \frac{\text{ADACmember : } \text{likesCars}}{\text{likesCars}} \]

Consistency w.r.t. to \( W \) alone is not enough.

Solution by Reiter: Use a theory that plays the role of a context or belief set. Check consistency against this context.

A formalization of this idea:

A default \( \delta = \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi} \) is applicable to a deductively closed set of formulae \( F \) w.r.t. belief set \( E \) (the context) iff \( \varphi \in F \) and \( \neg \psi_1 \notin E, \ldots, \neg \psi_n \notin E \) (each \( \psi_i \) is consistent with \( E \)).

Note that the concept “\( \delta \) is applicable to \( E \)” is so far a special case where \( E = F \).
Which context to use?

Observation:
If a default has been applied to a belief set $E$, a formula has been derived and is part of the knowledge base. Therefore it should be believed, i.e. become an element of belief set $E$.
On the other hand, $E$ should contain only formulae that can be derived from the axioms in $W$ by default application.

Definition 3.8 (Closure under a set of defaults w.r.t. a belief set)
Let $D$ be a set of defaults and $F$ a deductively closed set of formulae. $F$ is closed under $D$ w.r.t. belief set $E$ iff, for every default $\delta \in D$ that is applicable to $F$ w.r.t. belief set $E$, its consequent $\chi$ is also contained in $E$.

Lemma 3.9
Let $E' \subseteq E$ and $F$ be a set of formulae closed under some set of defaults $D$ w.r.t. $E'$. Then $F$ is closed under $D$ w.r.t. $E$.
Proof: exercise
Definition 3.10 ($\Lambda_T(E)$, extension)

Given $T = (W, D)$ and a set of formulae $E$. Let $\Lambda_T(E)$ be the smallest\(^3\) set of formulae that

- contains $W$,
- is closed under deduction, i.e. contains all conclusions, and
- is closed under $D$ w.r.t. $E$.

$E$ is an extension of $T$, iff $E = \Lambda_T(E)$.

Intuition:

- $\Lambda_T(E)$ contains all formulae that are sanctioned by $T$ w.r.t. $E$.
- $E$ is an extension, iff by the use of $E$ as a belief set, exactly the formulae in $E$ will be obtained from default application.

Observe: one first needs to guess $E$ and then check whether the fixed-point equation is fulfilled.

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\(^3\) i.e. has smallest number of elements
Are the two definitions of extension equivalent?
Are operational semantics and the original one the same?

**Theorem 3.11**

Let $T = (W, D)$ be a default theory. 
$E$ is an extension of $T$ (according Definition 3.5) iff $E = \Lambda_T(E)$.

Proof: blackboard
Minimality of Reiter’s extensions

Reiter’s characterization fulfills the desirable properties of an extension:

- includes the set $W$ of facts: $E$ includes $W$.
- is deductively closed: $E$ is deductively closed
- is closed under the application of the defaults: $E$ is closed under $D$ w.r.t. $E$

Claim: $E$ is minimal w.r.t. these properties.

Corollary 3.12 (Minimality of extension)

Let $E$ and $E'$ be two extensions of a default theory $T$. If $E' \subseteq E$, then $E' = E$.

Proof: If $E'$ is an extension and $E' \subseteq E$, then $E'$ is closed under $D$ w.r.t. $E$ (by Lemma 3.9).

By definition, $E = \Lambda_T(E) \subseteq E'$ and thus $E' = E$. 
Properties of extensions

**Theorem 3.13 (Consistency preservation)**
A default theory $T = (W, D)$ has an inconsistent extension iff $W$ is inconsistent.

Proof: Exercise

**Corollary 3.14**
If a default has an inconsistent extension $E$, then it is its only extension.

**Theorem 3.15**
Let $T = (W, D)$ be a default theory s.t. the set

$M = W \cup \{\psi_1 \land \cdots \land \psi_n \land \chi \mid \frac{\psi_1, \ldots, \psi_n}{\chi} \text{ is a default in } D\}$

is consistent. Then $T$ has exactly one extension.

Proof: blackboard
Nonmonotonic nature of Default logic

Nonmonotonic behaviour may appear when the default theory is changed!

Example 3.16 (Changing the defaults)
Let $T_{ex}(W, D)$ be a default theory with $W = \emptyset$ and $D = \{true:a\}$. $T_{ex}$ has exactly one extension: $E = Th(\{a\})$.

- $\delta_1 = \frac{true:b}{\neg b}$. Then $(W, D \cup \{\delta_1\})$ has no extensions.
- $\delta_2 = \frac{b:c}{c}$. Then $(W, D \cup \{\delta_2\})$ has $E$ as only extension.
- $\delta_3 = \frac{true: \neg a}{\neg a}$. Then $(W, D \cup \{\delta_3\})$ has two extensions: $E$ and $Th(\{\neg a\})$.
- $\delta_4 = \frac{a:b}{b}$. Then $(W, D \cup \{\delta_4\})$ has the two extensions: $Th(\{a, b\})$. 
Nonmonotonic nature of Default logic

Nonmonotonic behaviour may appear when the default theory is changed!

Example 3.17 (Changing the facts)
Let $T_{ex}(W, D)$ be a default theory with $W = \emptyset$ and $D = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ with

$$
\delta_1 = \frac{true : a, \neg c}{a}, \quad \delta_2 = \frac{a : b, \neg c}{b}, \quad \delta_3 = \frac{true : \neg a, c}{c}, \quad \delta_4 = \frac{d : e}{e}, \quad \delta_5 = \frac{f : g}{\neg g}
$$

$T_{ex}$ has two extensions: $E_1 = Th\{a, b\}$ and $E_2 = Th\{c\}$. Consider

- $W_1 = \{f\}$. ($W_1, D$) has no extensions.
- $W_2 = \{\neg a\}$. ($W_2, D$) has the only extension $Th\{\neg a, c\}$.
- $W_3 = \{\neg a, \neg c, d\}$. ($W_3, D$) the only new extension: $Th\{\neg a, \neg c, d, e\}$.
- $W_4 = \{d\}$. ($W_4, D$) the two extension: $Th(E_1 \cup \{d, e\})$ and $Th(E_2 \cup \{d, e\})$. 