Default theories are not semi-monotonic

Semi-monotonicity
A class of default theories, is semi-monotonic iff the addition of default rules never eliminates, but extends or adds, new extensions.

Consider the general default theories:

\[ T_1 = (\emptyset, D_1) \text{ with } D_1 = \left\{ \frac{\text{true} : \neg C}{D} \right\} \text{ and } \]
\[ T_2 = (\emptyset, D_2) \text{ with } D_2 = \left\{ \frac{\text{true} : B}{C}, \frac{\text{true} : \neg C}{D} \right\} \]

The theory \( T_1 \) has one extension \( E_1 = \text{Th}(\{D\}) \).
However, the only extension of \( T_2 \) is \( E_2 = \text{Th}(\{C\}) \).

\( E_1 \) fails to be an extension of \( T_2 \) since \( B \) is consistent with \( E_2 \), hence \( \frac{\text{true} : B}{C} \) is applicable and eliminating \( E_2 \) as possible extension.

Since we have \( D_1 \subseteq D_2 \), but \( E_1 \nsubseteq E_2 \), default logic is not semi-monotonic.
Theorem 3.21 (Semi-monotonicity)

Let \( T = (W, D) \) and \( T' = (W, D') \) be normal default theories s.t. \( D \subseteq D' \). Then each extension of \( T \) is contained in an extension of \( T' \).

Proof: blackboard

Theorem 3.22 (Orthogonality of extensions)

Let \( E \) and \( F \) be different extensions of a normal default theory \( T \). Then \( E \cup F \) is inconsistent.

Proof: blackboard
Limitations of normal default theories

Are normal default theories expressive enough to model common sense reasoning?

Statements such as:

- “Typically birds fly”
- “Assume the accused is innocent unless you know otherwise”

can be captured by normal defaults:

\[
\begin{align*}
\text{bird}(x) \Rightarrow \text{flies}(x) \\
\text{accused}(x) \Rightarrow \text{innocent}(x)
\end{align*}
\]

Often a default rule on its own is normal, but problems arise when several defaults have to interact in a theory.
Limitations of normal default theories—example

Consider the example of a normal default theory:

\[ T = \left( \{ \text{dropout}(\text{bill}) \}, \left\{ \frac{\text{dropout}(x) : \text{adult}(x)}{\text{adult}(x)}, \frac{\text{adult}(x) : \text{employed}(x)}{\text{employed}(x)} \right\} \right) \]

\( T \) has the single extension \( Th(\{ \text{dropout}(\text{bill}), \text{adult}(\text{bill}), \text{employed}(\text{bill}) \}) \). But it is counterintuitive to assume that Bill is employed!

How to prevent the application of the 2. default, if \( X \) is a dropout?

\[ \text{adult}(x) : \text{employed}(x) \land \neg \text{dropout}(x) \]

\[ \frac{\text{employed}(x)}{\text{employed}(x)} \]

But this is no longer a normal default!
Semi-normal Defaults

A default is a semi-normal default, if it has the form $\phi : \psi \wedge \chi$. A default theory $T = (W, D)$ is semi-normal, if all defaults in $D$ are semi-normal.

Do semi-normal default theories always have extensions? No.

Consider the example: $T = (W, D)$, with $W = \emptyset$ and

$$D = \left\{ \frac{true : \neg q \wedge p}{p}, \frac{true : \neg r \wedge q}{q}, \frac{true : \neg p \wedge r}{r} \right\}$$

Only some classes of restricted semi-normal theories do always have extensions.
Semi-normal Default Theories

Semi-normal default theories do not have . . .

- **Semi-monotonicity**
  
  Consider:
  
  \[ T = (W, D) \text{ with } W = \emptyset \text{ and } D = \left\{ \frac{\text{true}: \neg q \land p}{p} \right\} \]
  
  \[ T' = (W, D') \text{ with } W = \emptyset \text{ and } D' = \left\{ \frac{\text{true}: \neg q \land p}{p}, \frac{\text{true}: \neg r \land q}{q} \right\}. \]

  We have \( E = \text{Th}(\{p\}) \) and \( E = \text{Th}(\{q\}) \) as the extensions of the theories, but \( E \not\subseteq E' \).

- **Success of all processes**
  
  \( T' \) has a failed process.

- **Orthogonality of extensions**
  
  Consider:
  
  \[ T'' = (W, D) \text{ with } W = \emptyset \text{ and } D = \left\{ \frac{\text{true}: p \land q}{q}, \frac{\text{true}: \neg q \land \neg p}{\neg p} \right\}. \]

  \( T'' \) has two extensions: \( E_1 = \text{Th}(\{q\}) \) and \( E_2 = \text{Th}(\{\neg p\}) \), but \( E_1 \cup E_2 \) is consistent.
Reasoning in Default Logics

Classical reasoning problems of default theories are:

- deciding whether a default theory has an extension
- deciding whether a given formula is element of all extensions. ("cautious reasoning" or "skeptical reasoning")
- deciding whether a given formula is element of one extension. ("brave reasoning" or "credulous reasoning")

Choosing a different extension, may yield different consequences.
Example: reasoning in Default Logics

Example 3.23 (Reprise of Example 3.17)

Let $T_{ex} = (W, D)$ be a default theory with $W = \{d\}$ and $D = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ with

\[
\begin{align*}
\delta_1 &= \frac{\text{true} : a, \neg c}{a}, \\
\delta_2 &= \frac{a : b, \neg c}{b}, \\
\delta_3 &= \frac{\text{true} : \neg a, c}{c}, \\
\delta_4 &= \frac{d : e}{e}, \\
\delta_5 &= \frac{f : g}{\neg g}
\end{align*}
\]

$T_{ex}$ has two extensions: $E_1 = \text{Th}(\{d, e, a, b\})$ and $E_2 = \text{Th}(\{d, e, c\})$.

Formula $(d \land e)$ is a consequence for $T_{ex}$ under cautious reasoning.

Formula $c$ is a consequence for $T_{ex}$ under brave reasoning.
Recap on complexity classes

Complexity class with oracle admits the use of a subrouting “at no cost”.

Polynomial hierarchy: the classes $\Pi_k^P$, $\Sigma_k^P$ and $\Delta_k^P$ are defined as follows:

\[
P = \Sigma_0^P = \Pi_0^P = \Delta_0^P
\]

and for all $k \geq 0$:

\[
\Sigma_{k+1}^P = NP\Sigma_k^P \quad \Pi_{k+1}^P = \text{co-}\Sigma_{k+1}^P \quad \Delta_{k+1}^P = P\Sigma_k^P
\]

Note: $\Sigma_1^P = NP$, $\Pi_1^P = \text{co-NP}$ and $\Delta_{k+1}^P = P$. 
Complexity of reasoning in default theories

Typically, default reasoning is harder than classical reasoning.

Computability / complexity results for different classes of default theories:

- **FOL default theories**: undecidable
  (since classical reasoning is already undecidable)

  Reasoning in default theories is not even semi-decidable, since computing an extension requires FOL consistency tests which are semi-decidable.

- **normal default theories**: undecidable

- **propositional default theories**:
  - deciding existence of an extension: $\Sigma_2^P$-complete
  - brave reasoning is $\Sigma_2^P$-complete
  - cautious reasoning is $\Pi_2^P$-complete