TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Faculty of Computer Science Chair of Automata Theory

INTRODUCTION TO NONMONOTONIC REASONING

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Section 6

Nonmonotonic Inference Relations

Subsection 6.1

Inference relations

Introduction

We have discussed three formalisms that provide nonmonotonic reasoning:
Default logics, Autoepistemic logic, and Circumscription.
They provide useful inference relations.

In this chapter we take a more abstract view:

- What are the properties of nonmonotonic inference relations in general—independent of a particular formalism?
- How to compare the different approaches?

Inferences in the discussed formalisms:

- Circumscription uses minimal, i.e., preferred models, yielding a preferential inference relation.
- Default logic and autoepistemic logic use some kind of fixpoint construction.

What is the semantics of their inference relation?

Inference relations

Given knowledge about the world, when could a formula φ be reasonably concluded from a set of formulae M?

Let T be a set of first order sentences (the agent's knowledge).

- classical entailment
  The agent classically infers a formula φ, if φ holds in all models of T in which M holds.

But an agent's knowledge might be incomplete, so nonmonotonic (/defeasible) inference relations are interesting. For example:

- under circumscription,
  the agent might infer φ from M if φ holds in every minimal model of T
- in default logic (if the agent's knowledge is given also by a set of default rules D), the agent might infer φ from M, if φ is in every extension of the default theory $(\{M\}, D)$.

Inference relations can be modeled as binary relations on (sets of formulae of) a logic $\mathcal{L}$. We denote nonmonotonic inference relations by $\mid\!\sim$.

Which binary relations on $\mathcal{L}$ are (non-monotonic) inference relations?

For example: inference relation for default logic

In default logic the inference relation could be defined as:
Given a countable set of defaults D.

$$W \mid\!\sim \varphi \text{ iff } \varphi \text{ is included in all extensions of } (W, D).$$

or alternatively

$$W \mid\!\sim \varphi \text{ iff } \varphi \text{ is included in some extension of } (W, D).$$

The setting considered

In this chapter we consider:

- propositional logic
- inference relation $\mid\!\sim$
- the inference operation C defined (for a given set of formulae M) as:

$$C(M) = \{\varphi \mid M \mid\!\sim \varphi\}.$$

We will use the inference relation $\mid\!\sim$ and the inference operation $C(M)$ interchangeably. E.g.: '$M \mid\!\sim \varphi$ if $\varphi \in M$ can be formulated as '$M \subseteq C(M)$'.

The properties discussed in the following also hold for the classical inference relation $\vdash$.
(But not all properties of $\vdash$ hold for nonmonotonic reasoning.)

Subsection 6.2

Basic properties: pure conditions

Pure conditions

Consider the following properties of an inference operation C:

- $M \subseteq C(M)$ — Inclusion
- $C(M) = C(C(M))$ — Idempotence
- $M \subseteq N \subseteq C(M)$ implies $C(N) \subseteq C(M)$ — Cut
- $M \subseteq N \subseteq C(M)$ implies $C(M) \subseteq C(N)$ — Cautious Monotony
- $M \subseteq N \subseteq C(M)$ implies $C(M) = C(N)$ — Cumulativity
- $M \subseteq N$ implies $C(M) \subseteq C(N)$ — Monotony

These are called pure conditions, since they do not refer to any features of the underlying logic.

Intuition of the pure conditions

- Inclusion
  requires that the inference operation extends the set of formulae
- Idempotence
  requires that, after having applied the inference operation, another application does not add new formulae.
- Cut
  ensures that if the information in M is expanded by some proposition included in the closure $C(M)$, then no new conclusions are obtained.
- Cautious Monotony (converse of Cut)
  the addition of a lemma does not decrease the set of conclusions.
- Cumulativity (Cut and Cautious Monotony combined)
  ensures that lemmas can be safely used without affecting the supported conclusions.
- Monotony
  an extended set of premises gives an extended set of conclusions.

Cumulative inference relation

An inference relation is called a cumulative inference relation if it satisfies

- Inclusion,
- Idempotence,
- Cut,
- Cautious Monotony, and
- Cumulativity.

What is the minimal set of conditions for an inference relation to be cumulative?

Theorem 6.1

1. *Cut and Inclusion imply Idempotence.*
2. *Cautious monotony and Idempotence imply Cut.*

Proof: blackboard

Since Cumulativity implies Cautious Monotony and also Cut, the properties Cumulativity and Inclusion are sufficient to obtain a cumulative inference relation.

Subsection 6.3

Basic properties: interaction with logical connectives

Basic properties linking classical and nonmonotonic logic

Note: In the following we refer to propositional logic.

Properties linking classical and nonmonotonic logic:

- $Th(M) \subseteq C(M)$ — Supraclassicality
- $Th(C(M)) = C(M)$ — Left Absorption
- $C(Th(M)) = C(M)$ — Right Absorption
- $Th(C(M)) = C(M) = C(Th(M))$ — (Full) Absorption

Connection of Supraclassicality to pure and basic properties

Supraclassicality ensures that the classical consequences follow nonmonotonically, too, i.e. nonmonotonic inference should support more conclusions!

Theorem 6.2

Let C be a supraclassical inference relation.

1. *If C satisfies Idempotence, then it satisfies Left Absorption.*
2. *If C is Cumulative, then it satisfies Full Absorption.*

Proof: blackboard

Absorption properties establish some more properties

The following properties follow from the absorption properties and link nonmonotonic inference and logical connectives.[1]

- $M \mid\!\sim \varphi$ and $M \mid\!\sim \psi$ implies $M \mid\!\sim \varphi \wedge \psi$ — Right And
- $M \mid\!\sim \varphi$ and $\{\varphi\} \vdash \psi$ implies $M \mid\!\sim \psi$ — Right Weakening
- $M \mid\!\sim \varphi$ and $Th(M) = Th(N)$ implies $N \mid\!\sim \varphi$ — Left Logical Equivalence

A note on Right Weakening:

If $\mid\!\sim$ would have been used instead of $\vdash$, then a form of transitivity would have been obtained.

[1] Since the names of these properties are more intuitive when displayed as a relation instead of an operation, we use this version.

Transitivity of nonmontonic inference relations?
—not if superclassicality should hold!

Transitivity together with Supraclassicality yield a form of monotony!

Theorem 6.3
Let $\mid\!\sim$ be transitive and supraclassical. Then $\{\varphi\} \mid\!\sim \chi$ implies $\{\varphi \wedge \psi\} \mid\!\sim \chi$.

Proof: blackboard

$\rightsquigarrow$ If Supraclassicality, but not Monotony should hold, then transitivity must be given up.

Distribution

Another important property of inference relations:

- $C(M) \cap C(N) \subseteq C\big(Th(M) \cap Th(N)\big)$ Distribution

An inference relation that satisfies Distributivity is called distributive.

Lemma 6.4

If C satisfies Absorption, then Distribution is equivalent to the following conditions:

1. *If $M = Th(M)$ and $N = Th(N)$ then $C(M) \cap C(N) \subseteq C(M \cap N)$*
2. $C(T \cup M) \cap C(T \cup N) \subseteq C\big(T \cup (Th(M) \cap Th(N))\big)$.

Proof: exercise

Properties that follow from Distribution

- $C(M \cup \{\varphi\}) \cap C(M \cup \{\psi\}) \subseteq C(M \cup \{\varphi \vee \psi\})$ Left Or
- $C(M \cup \{\varphi\}) \cap C(M \cup \{\neg\varphi\}) \subseteq C(M)$ Proof by Cases

Theorem 6.5

If C satisfies Distribution, Supraclassicality and Absorption, then C satisfies Left Or and Proof by Cases.

Proof: blackboard

Subsection 6.4

Properties of inference operations in Default Logic

Definition 6.6 (Inference operations in default logic)

Let D be a countable set of default rules.

- Skeptical inference operation $C_{D,Ske}(M)$ is defined as the intersection of all extensions of the default theory $T = (M, D)$.
- Credulous inference operation $C_{D,Cre}(M)$ is defined as the union of all extensions of the default theory $T = (M, D)$.

On the credulous inference operation $C_{D,Cre}(M)$:

- tends to be irregular and violates most properties
- satisfies Left Logical Equivalence
  ($M \mid\sim \varphi$ and $Th(M) = Th(N)$ implies $N \mid\sim \varphi$)
- satisfies Right Weakening
  ($M \mid\sim \varphi$ and $\{\varphi\} \vdash \psi$ implies $M \mid\sim \psi$).

We concentrate in the following on skeptical inference operation $C_{D,Ske}(M)$.

A positive result for skeptical inference

Theorem 6.7

The skeptical inference operation $C_{D,Ske}(M)$ *of default logic satisfies*

- *Cut and*
- *Absorption.*

Proof: blackboard

Skeptical inference violates Cumulativity

Skeptical inference operation $C_{D,Ske}(M)$ does satisfy Cut, but it does not satisfy Cumulativity.

Example 6.8 (Counter example for Cumulativity)

To see that "$M \subseteq N \subseteq C(M)$ implies $C(M) = C(N)$" does not need to hold, consider $T = (W, D)$ with $W = \emptyset$ and

$$D = \left\{ \frac{true \ : \ a}{a}, \frac{a \vee b \ : \ \neg a}{\neg a} \right\}.$$

The only extension of T is $Th(\{a\})$.
Obviously, a is included in all extensions of T, i.e., $a \in C_{D,Ske}(W)$.
From $(a \vee b) \in Th(\{a\})$, we get $(a \vee b) \in C(W)$.

If we use $W' = W \cup \{a \vee b\}$, then the default theory $T' = (W', D)$ has two extensions: $Th(\{a\})$ and $Th(\{\neg a, b\})$. Thus $a \notin C_{D,Ske}(W \cup \{a \vee b\})$.

Skeptical inference violates Distributivity

Skeptical inference operation $C_{D,Ske}(M)$ does not satisfy Distributivity.

Example 6.9 (Counter example for Distributivity)

To see that "$C(M) \cap C(N) \subseteq C\big(Th(M) \cap Th(N)\big)$" does not need to hold for skeptical inference, consider

$$D = \left\{ \frac{\varphi \; : \; \chi}{\chi}, \; \frac{\neg\varphi \; : \; \chi}{\chi} \right\}.$$

Then $\chi \in C(\{\varphi\})$ and $\chi \in C(\{\neg\varphi\})$, but $\chi \notin C_{D,Ske}\big(Th(\{\varphi\}) \cap Th(\{\neg\varphi\})\big)$.

Subsection 6.5

Inference relations based on preferential models

On preferential models

The idea of preferential models is to generalize the concept of minimal models that were used for Circumscription.

Reasoning under preferential models semantics does no longer consider all models to compute consequences, but only preferred ones.

In the following we abstract from the underlying logic and use $\mathcal{L}$ and its elements (‘propositions’).

Definition 6.10 (preferential model structure)

A preferential model structure is a triple $(MS, \models, <)$, where

- MS is a set of models.
- $\models \ \subseteq MS \times \mathcal{L}$ is a relation between models and propositions in $\mathcal{L}$ and is called satisfaction relation of the structure.
- $< \ \subseteq MS \times MS$ is a relation on MS and is called the preference relation.

Let $m \in MS$ and $L \subseteq \mathcal{L}$.

A model m preferentially satisfies L (denoted $m \models_< L$) iff
$m \models L$ and there is no model $m' \in MS$ s.t. $m' < m$ and $m' \models L$.

We call m a preferential model of L.

The intuition for the

- satisfaction relation $\models$ is that it states which propositions are satisfied by which models.
- preference relation $<$ is that it states which models are preferred over which other models.

Definition 6.11 ($\models_<$, $C_<$)

Based on preferential models we define an inference relation $\mathrel{|\!\!\sim}_<$ determined by a preferential model structure $(MS, \models, <)$ as:

$$L \mathrel{|\!\!\sim}_< x \text{ iff for all } m \in MS \text{ holds: } m \models_< L \text{ implies } m \models x.$$

The inference operation $C_<$ determined by a preferential model structure $(MS, \models, <)$ is then:

$$C_<(L) = \{x \in L \mid \text{ for all } m \in MS, m \models_< L \text{ implies } m \models x\}.$$

Intuition:
Formula x follows nonmonotonically from a set of formulae L if it is satisfied by all preferential models of L.

Theorem 6.12

Every preferential model structure satisfies

- *Inclusion,*
- *Idempotence and*
- *Cut.*

Proof: blackboard

Pure conditions of preferential model structures

Preferential model structures do not satisfy Cumulativity.

Example 6.13

Let MS be the infinite set $\{m_1, m_2, \ldots\}$, and define $m_i < m_j$ iff $j < i$.
Let also x, y, z be elements of the underlying language $\mathcal{L}$ such that:

$$m_i \models x \text{ for all } i > 0$$
$$m_i \not\models y \text{ for all } i > 0$$
$$m_i \models z \text{ iff } i = 1$$

There is no minimal model satisfying x, therefore $\{x\} \mid\!\sim y$ and $\{x\} \mid\!\sim z$.
But m_1 is the minimal model satisfying $\{x, z\}$ and $m_1 \not\models y$.
So, $\{x, z\} \mid\!\sim y$ is false and Cautious Monotony and thus Cumulativity is violated.